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Corner Transfer Matrices of the Chiral Potts Model. II. The Triangular Lattice

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We consider a two-dimensional edge-interaction model satisfying the star-triangle relations. For the triangular lattice, the corner transfer matrices are functions of three rapidities: we show that they possess various factorization properties and satisfy certain equations. We indicate how these equations can be solved for the Ising model. We then consider the three-state chiral Potts model and obtain low-temperature solutions to the equations. The conjectured formula for the order parameter (the spontaneous magnetization) is verified to one more order in a series expansion.

KEY WORDS: Statistical mechanics; lattice models; chiral Potts model; corner transfer matrices.

1. INTRODUCTION

The "chiral Potts" model continues to be the subject of much study in two-dimensional lattice statistical mechanics. It was first formulated as a one-plus-one dimensional integrable system[1,3]; then in 1987 Au-Yang, McCoy, and Perk began the search for the corresponding two-dimensional lattice model, having an N-state spin at each site and satisfying the star-triangle of "Yang–Baxter" relations. They found the $N=3$ solution[4,5] and the self-dual $N=4$, 5 solutions.[6,7]

In 1988, the general-N solution was found by Baxter, Perk, and Au-Yang (the solution is stated in ref. 8, the proof is partly given in the Appendix of ref. 9 and completed in ref. 10). When $N=2$ it reduces to the Ising model, while if one approaches criticality in an appropriate manner it becomes the Fateev–Zamolodchikov model.[11]

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As with any "Z-invariant" model (i.e., one that satisfies the star-triangle relations), each site can be regarded as lying at the intersection of two rapidity lines. The Boltzmann weights w of that site are functions of the corresponding two rapidities p and q. All previous Z-invariant models had the "difference" property that p and q could be chosen so that the w depend on them only via their difference, i.e., $w = w_{pq} = w(p-q)$. However, the chiral Potts model is remarkable in that it does *not* have this difference property.

Another (related) distinction is that for the earlier models varying a rapidity causes the Boltzmann weights to trace out an algebraic curve of genus 0 or 1. They can therefore be parametrized by a uniformizing substitution as single-valued functions of one complex variable; indeed this substitution is precisely the one that manifests the difference property. For the chiral Potts model the genus is higher than one and there is no such simple substitution.

Because the model is Z-invariant, we expect it to be solvable, in the sense that the bulk free energy, correlation length (or mass gap), interfacial tension, and single-spin order parameters (the analogs of the Ising model spontaneous magnetization) should be exactly calculable. Unfortunately, the absence of the difference property and of a one-variable uniformizing substitution makes it difficult to adapt the previous methods employed for the Ising, six-vertex, and eight-vertex models.[12] Even so, the free energy was obtained (albeit in a rather complicated form) in 1988.[13]

Equations have been obtained for the eigenvalues of the usual row-to-row transfer matrix.[14–20] They have been solved to obtain an alternative expression for the free energy.[21,22] It should be possible to also use them to obtain the correlation length and interfacial tension. Numerical studies have been made for small lattices.[23,24]

The equations simplify dramatically in the "superintegrable" case. [With the notation of (2.81), this arises when the vertical rapidities alternately take the values p and p', where $a_{p'}$, $b_{p'}$, $c_{p'}$, $d_{p'} = b_p$, a_p, d_p, c_p. A specialization of this occurs when the system is homogeneous and $a_p = b_p$, $c_p = d_p$.] This case has been much studied and the correlation length and interfacial tension evaluated.[25,26] (Intriguingly, it turns that there is a related Hamiltonian, wioth fixed spin boundary conditions, which has a simple direct-sum eigenvalue spectrum.[27,28])

From the point of view of statistical mechanics the superintegrable case is unphysical in that the Boltzmann weights are not positive or even real. However, the corresponding $(1+1)$-dimensional integrable system has a Hermitian Hamiltonian: it exhibits level crossing[29–31] and has been proposed as a model of high-T_c superconductivity.[32]

The N-state chiral Potts model has also been extended to an N^{n-1}-

state model associated with the algebra $U_q(sl(n))$.[33] Very recently it has been noted that this can be regarded as a model on the three-dimensional simple cubic lattice consisting of n square lattice layers.[34] Further, this model is a generalization of the three-dimensional Zamolodchikov model,[35-37] reducing to it when $N=2$.

An outstanding problem remains the calculation of the single-site order parameters. For previous models, such as the eight-vertex model, these have been calculated using corner transfer matrices (CTMs).[39,40,12,41] These can be defined for any planar model. Here we consider the trangular lattice (since in some ways this most fuly exhibits the interrelations of the rapidity dependences coming from the star-triangle relation), extending our previous discussion for the square lattice.[38] In Section 2 we define the CTMs $A_1,...,A_6$ and related matrices $F_1,...,F_6$. They satisfy Eqs. (2.3)–(2.4), which define them to within irrelevant scale and similarity transformations. Particularly important is the matrix M of eigenvalues of the product $A_1 \cdots A_6$, defined by

$$A_1 \cdots A_6 = B^{-1} M B \tag{1.1}$$

where M is diagonal and B is invertible. The order parameters can be expressed in terms of M, as is done in (2.13).

We then go on to show that for a Z-invariant model the CTMs are functions of three rapidities p, q, r and possess the factorization properties (2.36), (2.47). They satisfy Eqs. (2.43)–(2.45), or equivalently (2.52), (2.53).

For the eight-vertex and other previous models, we can also these equations exactly for the diagonalized forms of $A_1,...,A_6$ and M, and hence obtain the order parameter. In Section 3 we indicate how to do this for the $N=2$ Ising case of the chiral Potts model. A vital ingredient is the difference property: together with the factorization property it ensures that $A_1,...,A_6$ are basically exponential functions of the rapidities, or more precisely that (2.36) implies (3.24) and (3.25).

For $N \geqslant 3$ we do not have the difference property. The model is still integrable, so we still expect the order parameter to be exactly calculable. In fact there is an intriguingly simple conjecture for the order parameter: if a is the spin at a central site, taking the values $0,..., N-1$, then for $0 \leqslant j \leqslant N$

$$\langle \omega^{ja} \rangle = (1-k'^2)^{j(N-j)/2N^2} \tag{1.2}$$

[Eq. (3.13) of ref. 2, Eq. (1.20) of ref. 15, Eq. (15) of ref. 42, β and λ therein being the k' of this paper; the system is to be ferromagnetically ordered, which implies $0 < k' < 1$]. For $N=3$ this conjecture was verified by Howes *et al.*[2] to order k'^{13}.

Can we prove this? In Section 4 we consider in some detail the $N=3$ case, using the hyperelliptic parametrization[43] with nome x given by (4.1). We develop leading-order solutions of the CTM equations at low temperatures, then extending them to next and higher orders in the variable x, which is of order k'^2. We particularly consider the eigenvalues of M, calculating them to order up to but not including x^8. (Because we considered the triangular lattice, our expansions were in powers of $x^{1/3}$, so we actually worked with series to order $x^{23/3}$. Only the first 17 eigenvalues contribute to this order.)

The results for M are given in (4.61) and (4.62). They agree with the conjecture (1.2) and extend the verification one more order, from x^6 to x^7 (strictly, from $x^{13/2}$ to $x^{23/3}$). Unlike the previously solved cases, the eigenvalues of M are *not* just simple powers of some common variable x, and at this stage it is not clear (at least to the author) how to proceed. The hope is that these results will provide a testing ground for an analytic investigation of the equations, and that ultimately a derivation of (1.2) will materialize, along with an understanding of the CTMs of the chiral Potts model.

2. TRIANGULAR LATTICE CTMs

2.1. Basic Definitions and Equations

For the triangular lattice, the CTM equations for an isotropic and reflection-symmetric model were written down by Baxter and Tsang.[44] The chiral Potts model is by definition reflection asymmetric, and contains no isotropic case, so here we have to extend the equations to anisotropic and asymmetric models. This is straightforward, if a little cumbersome.

Consider a triangular lattice $\mathscr{L}$ of finite extent. On each site i place a "spin" σ_i with values $0, 1, \ldots, N-1$. With each left-pointing triangle (i, j, k) associate a Boltzmann weight $\Omega_1(\sigma_i, \sigma_j, \sigma_k)$, and with each right-pointing triangle a weight $\Omega_4(\sigma_i, \sigma_j, \sigma_k)$, as in Fig. 1. Then the partition function is

$$Z = \sum_{\sigma_1, \sigma_2, \ldots} \prod \Omega_1(\sigma_i, \sigma_j, \sigma_k) \prod \Omega_4(\sigma_i, \sigma_j, \sigma_k) \tag{2.1}$$

where the first (second) product is over all left-point (right-pointing) trianghes (i, j, k), and the sum is over all configurations of all the spins.

Now consider the lattice segments of r rows shown in Fig. 2. Let λ denote all the spins on the lower edge, including the spin a at the left; similarly, let μ denote all the upper spins, including b. Let $A^{(2)}_{\lambda\mu}$ be the

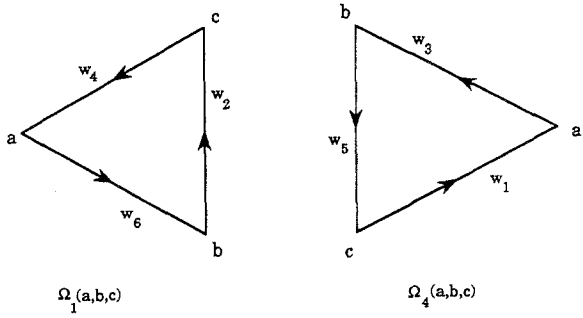

Fig. 1. The triangle weights.

partition function of segment 2(a), summed over all interior spins; and let $F_{\lambda\mu}^{(21)}$ be the corresponding partition function for segment 2(b).

We can regard a as a function E of λ (the end spin of the set), so that $a = E(\lambda)$. Note that on 2(a) the leftmost spins a and b in λ and μ are necessarily the same: we extend the definition of $A_{\lambda\mu}^{(2)}$ be defining it to be zero if $a \neq b$, i.e., if $E(\lambda) \neq E(\mu)$.

Now let A_2 be the matrix with element $A_{\lambda\mu}^{(2)}$ in position (λ, μ); similarly for F_2. Define $A_3, \ldots, A_6, A_1$ and $F_3, \ldots, F_6, F_1$ similarly for the other segments obtained by rotation.

For any matrix M with elements $M_{\lambda\mu}$, we define an associated matrix M^* with elements

$$\begin{aligned} (M^*)_{\lambda\mu} &= M_{\lambda\mu} \quad \text{if} \quad E(\lambda) = E(\mu) \\ &= 0 \quad \text{if} \quad E(\lambda) \neq E(\mu) \end{aligned} \tag{2.2}$$

This means that M^* is a block-diagonal matrix (at least after a rearrangement of the rows and colums), there being N blocks, each correponding to a particular value of the end spin $a = E(\lambda) = E(\mu)$. From now on, when we say that any matrix L is "block-diagonal," we mean that it has this

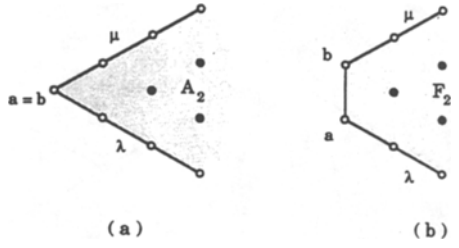

Fig. 2. Lattice segments corresponding to A_2 and F_2; the spins on the right-hand boundary are fixed to be zero.

structure, i.e., $L^* = L$. By its "diagonal block a" we mean the set of elements (λ, μ) such that $E(\lambda) = E(\mu) = a$.

In particular, the matrices A_i are block-diagonal: $A_i^* = A_i$. Hence, for instance, if $N = 3$, we can arrange the rows and columns so that

$$A_i = \begin{pmatrix} A_i^{(0)} & 0 & 0 \\ 0 & A_i^{(1)} & 0 \\ 0 & 0 & A_i^{(2)} \end{pmatrix}$$

where $A_i^{(a)}$ is the diagonal block a of A_i, represented in Fig. 2a. We shall sometimes write this simply as A_i^a.

With these definitions, consider the (λ, μ) elements of the matrix products $(F_1 A_2 A_3 F_4)^*$ and $A_1 A_2 A_3 A_4$. They are the partition functions of the lattices shown in Fig. 3, with the same external spins (λ and μ on the left-hand edges). (Matrix multiplication is equivalent to summing over the spins on the filled circles.) Using translation invariance, these lattices have the same external spins on the left-hand edges, and differ only in the position of the top right boundary. In the limit of r large, we expect this boundary difference to contribute only a scalar factor ξ_1 that is independent of λ and μ. Hence we obtain the matrix equation

$$(F_1 A_2 A_3 F_4)^* = \xi_1 A_1 A_2 A_3 A_4 \tag{2.3}$$

Similarly, considering the lattices in Fig. 4, we obtain the equation

$$\sum_{\nu\nu'} \Omega_1(a, c, b) F^{(6)}_{\lambda\nu} A^{(1)}_{\nu\nu'} F^{(2)}_{\nu'\mu} = \eta_1 (A_6 F_1 A_2)_{\lambda\mu} \tag{2.4}$$

where $a = E(\lambda)$, $b = E(\mu)$, $c = E(\nu) = E(\nu')$.

Define other (rotated) triangular weight functions Ω_2, Ω_3, Ω_5, Ω_6 by

$$\Omega_i(a, b, c) = \Omega_{i-2}(c, a, b) = \Omega_{i+6}(a, b, c) \tag{2.5}$$

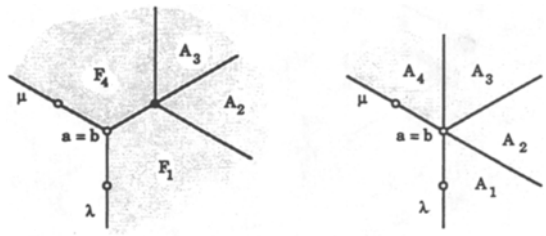

Fig. 3. The lattice segments $(F_1 A_2 A_3 F_4)^*_{\lambda\mu}$ and $(A_1 A_2 A_3 A_4)_{\lambda\mu}$.

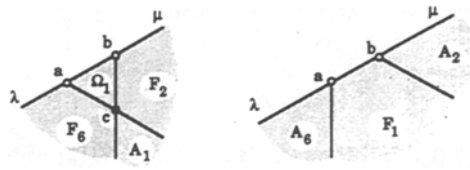

Fig. 4. The lattice segments of the LHS of (2.4) and $(A_6F_1A_2)_{\lambda\mu}$.

Then five other equations can be obtained from each of (2.3) and (2.4) by cyclically permuting the indices 1,..., 6, which is equivalent to rotating the lattice.

Let

$$Z^{(0)} = \text{Trace } A_1 \cdots A_6$$
$$Z_1^{(1)} = \text{Trace } F_1 A_2 A_3 F_4 A_5 A_6 \tag{2.6}$$
$$Z_1^{(2)} = \sum_{\lambda\lambda'\mu\mu'\nu\nu'} \Omega_1(a,b,c) A^{(5)}_{\lambda\lambda'} F^{(6)}_{\lambda'\mu} A^{(1)}_{\mu\mu'} F^{(2)}_{\mu'\nu} A^{(3)}_{\nu\nu'} F^{(4)}_{\nu'\lambda}$$

These are partition functions of lattices that have been "sliced" like a cake into six or seven pieces, as indicated in Fig. 5. Define $Z_2^{(1)},..., Z_6^{(2)}$ similarly, by cyclicaly permuting the indices 1,..., 6. Then we have the totational symmetries

$$Z_i^1 = Z_{i+3}^1, \qquad Z_i^2 = Z_{i+2}^2, \qquad \forall i \tag{2.7}$$

and from Eqs. (2.3), (2.4)

$$\xi_i = Z_i^{(1)}/Z^{(0)}, \qquad \eta_i = Z_i^{(2)}/Z_i^{(1)} \tag{2.8}$$

Let $\mathcal{N}$ be the number of sites in one of these lattices, with partition function Z. Then in the large-size limit we expect there to be a "partition function per site" κ such that $Z = B\kappa^{\mathcal{N}}$, where the boundary factor B itself

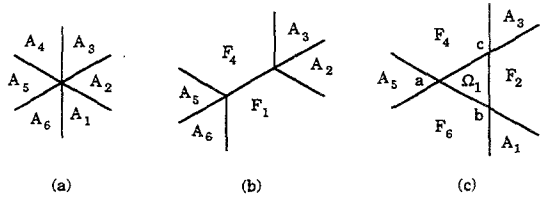

Fig. 5. The lattices with partition functions given by (2.6).

factors into contributions from the various parts of the boundary. It follows that

$$\kappa = Z^{(0)} Z_1^{(2)} Z_2^{(2)} / [Z_2^{(1)} / [Z_1^{(1)} Z_3^{(1)} Z_5^{(1)}] \tag{2.9}$$

In fact, if we regard the matrices A_i, F_i as arbitrary and choose them so as to maximize (2.9), then we obtain Eqs. (2.3) and (2.4). Thus, at least in this sense, the equations are derivable from a variational principle and are the extension to the triangular lattice of the equations for the square lattice.[39] (They can probably also be obtained from the variational principle for maximum eigenvalue of the usual row-to-row transfer matrix, as was originally done for the square lattice monomer–dimer problem.[45]) Using (2.8), we see that

$$\kappa = \eta_{i+1}\eta_{i+2}/\xi_i = \eta_{i+1}\eta_{i+2}/\xi_{i+3} \tag{2.10}$$

To obtain an averaged single-site property, such as the magnetization, define a diagonal matrix S with elements

$$\begin{aligned} S_{\lambda\mu} &= s(a) \qquad \text{if} \quad \lambda = \mu \\ &= 0 \qquad\quad \text{if} \quad \lambda \neq \mu \end{aligned} \tag{2.11}$$

where s is an arbitrary function and $a = E(\lambda)$. Then for the center spin a in Fig. 5a, the average of $s(a)$ is

$$\langle s(a) \rangle = \frac{\text{Trace } S A_1 \cdots A_6}{\text{Trace } A_1 \cdots A_6} \tag{2.12}$$

The matrix M is defined by (1.1) to be the diagonalized form of $A_1, \ldots, A_6$. Since $A_1, \ldots, A_6$ are block-diagonal matrices, so are P and M. They all commute with S, so (2.12) can equivalently be written as

$$\langle s(a) \rangle = \frac{\text{Trace } S M}{\text{Trace } M} \tag{2.13}$$

2.2. Truncated Matrices and Their Solution

Equations (2.3) and (2.4) are true only if the matrices are infinite dimensional. However, if we regard them as derived from the requirement that (2.9) be maximized, then we can take the A_i, F_i all to be finite-dimensional n by n matrices. This will not give the true value of κ, but will give a good self-consistent approximation to it. In fact, we know[46,47,40] [see Eq. (14.1.17) of ref. 12] that small values of n can give κ correctly to quite a large number of terms in a series expansion.

Equations (2.3) and (2.4) are unchanged by the transformation $A_i \to L_{i-1}^{-1} A_i L_i$, $F_i \to L_{i-1}^{-1} F_i L_i$, provided each L_i is block-diagonal and $L_{i+6} = L_i$. We can use this to ensure that $A_1, \ldots, A_6$ are all diagonal, and this is usually a convenient basis in which to work. Then the indices λ, μ, ν can no longer be identified as spin sets: they are simply row and column indices, usually taking the values $1, \ldots, n$. However, because of the block-diagonal property of the A_i and L_i, there is still a function $E(\lambda)$ that takes the values $0, \ldots, N-1$ and can be thought of as the "end spin" associated with row (or column) λ. Let M_a be the size of the block for which $E(\lambda) = a$. Then $n = m_0 + \cdots + m_{N-1}$.

The technique for solving the equations is similar to that for the square lattice.[39,40] For $c = 0, \ldots, N-1$, let U_i^c be the n by n matrix with elements

$$(U_i^c)_{\lambda\mu} = \Omega_{i+1}(a, b, c) F_{\lambda\mu}^{(i)} \tag{2.14}$$

where $a = E(\lambda)$ and $b = E(\mu)$ and indices i are to be interpreted modulo 6, so $\Omega_7 = \Omega_1$. Also, let P_i^b be the n by m_b matrix whose columns are the columns μ of $A_{i-1} F_i A_{i+1}$ for which $E(\mu) = b$. Thus, to within a reordering of the columns,

$$A_{i-1} F_i A_{i+1} = (P_i^0, P_i^1, \ldots, P_i^{N-1}) \tag{2.15}$$

Similarly, let Q_i^a be the m_a by n matrix whose rows are the rows λ of $A_{i-1} F_i A_{i+1}$ for which $E(\lambda) = a$. Then, to within a reordering of the rows,

$$A_{i-1} F_i A_{i+1} = \begin{pmatrix} Q_i^0 \\ Q_i^1 \\ \cdots \\ Q_i^{N-1} \end{pmatrix} \tag{2.16}$$

and (2.4) and its rotated analogs can be written as

$$U_{i-1}^b P_{i+1}^b = \eta_i P_i^b [A_{i+2}]^b \tag{2.17}$$

or alternatively as

$$Q_{i-1}^a U_{i+1}^a = \eta_i [A_{i-2}]^a Q_i^a \tag{2.18}$$

where $[A_i]^a$ is the m_a black a of the block-diagonal matrix A_i. Equation (2.4) gives

$$Q_i^a P_{i+3}^a = \xi_i [A_{i-1} A_i \cdots A_{i+4}]^a \tag{2.19}$$

Suppose the system is in a feromagnetically ordered phase in which the ground state has all spins equal to zero. Normalize the weights so that $\Omega_i(0, 0, 0) = 1$. Then the smallest truncation is to take $m = m_0 = 1$, $m_1 = \cdots = m_{N-1} = 0$, so that the A_i, F_i, U_i^0, P_i^0, Q_i^0 are one-by-one matrices, which can all be chosen to be unity. Then $\xi_i = \eta_i = 1$.

Given an initial guess at the matrices $F_1, \ldots, F_6$, the following iterative procedure appears to converge [at least at sufficiently low temperatures, i.e. when the $\Omega_i(a, b, c)$ other than $\Omega_i(0, 0, 0)$ are sufficiently small].

(i) Calculate the U_i^c from (2.14).

(ii) Solve (2.17) for the P_i^b and $\eta_i A_i^b$. This can be done column by column and is equivalent to diagonalizing the matrix product $U_1^b \cdots U_6^b$. The eigenvalues are the diagonal elements of $\eta_1 \cdots \eta_6 A_1^b \cdots A_6^b$, and it is convenient to arrange them in numerically decreasing order. Thus the largest is in row and column 1.

The columns of the P_i^b are eigenvectors of this diagonalization problem, and the diagonal elements of $\eta_i A_i^b$ are corresponding eigenvalues. The η_i can be fixed by requiring $(A_i)_{11} = 1$ and one is free to choose a convenient normalization of each column of each matrix P_i^b. The elements of the diagonal matrices A_i are then determined.

Taking $b = 0, \ldots, N-1$, there are altogether nN eigenvalues: one *selects* the n of these (m_0 from $b = 0, \ldots, m_N - 1$ from $b = N-1$) that correspond to the initial guess. To maximize (2.9), it seems these should be the n numerically largest eigenvalues of $U_1^b \cdots U_6^b$.

(iii) Similarly, solve (2.18) for the relevant rows of the Q_i^a: the rows are the associated left eigenvectors of the diagonalization problem in (ii), and their normalization is fixed by the orthonormality condition (2.19). One can choose $\xi_i = 1$, or else choose it to ensure $(F_i)_{11} = 1$.

(iv) Arrange the columns on the RHS of (2.15) to correspond to those on the LHS; similarly for the rows in (2.16). Calculate the elements of F_i from either of these equations. The choice can be significant: provided the eigenvalues of $U_1^b \cdots U_6^b$ are arranged in numerically decreasing order, it seems that the upper right elements should be calculated from (2.15), the lower left from (2.16) (and the diagonal elements from either).

Now one repeats this procedure until it has sufficiently converged.

This procedure also gives the clue how for to go from an n by n truncation to a larger one: one simply keeps more than n of the eigenvalues and associated eigenvectors in stages (ii) and (iii). The best choice seems to be to keep those corresponding to the next largest eigenvalue of $U_1^b \cdots U_6^b$ (or set of such eigenvalues if there are more than one of the same order of

magnitude). If one keeps n' eigenvalues, where $n' > n$, then at stage (iv), F_i is calculated to be an n' by n' matrix, but the bottom right $n' - n$ by $n' - n$ block is undetermined and is taken to be zero. For instance, for $n = 3$ and $n' = 5$,

$$F_i = \begin{pmatrix} u & u & u & u & u \\ d & u & u & u & u \\ d & d & u & u & u \\ d & d & d & 0 & 0 \\ d & d & d & 0 & 0 \end{pmatrix} \tag{2.20}$$

where the term denoted u are calculated from (2.15) and those denoted d from (2.16). One has to perform one more iteration to calculate the four lower right elements. [They can also be obtained from (2.4), solving for these elements on the RHS.]

This procedure is carried out explicitly in Appendix B for the three-state chiral Potts model, to leading order in a low-temperature perturbation expansion.

Apart from the scalar normalization factor $\eta_1 \cdots \eta_6$, the diagonal elements of the matrix product $A_1 \cdots A_6$ are eigenvalues of $U_1^c \cdots U_6^c$, for $c = 0, \ldots, N-1$. That this is so is not surprising, for from Fig. 6, U_i^c is itself a corner transfer matrix, but of larger dimensionality than A_i. Ultimately we are interested in the limit $n \to \infty$, when the matrices are infinite dimensional and the equations give κ and $\langle s(a) \rangle$ exactly. Then U_i^c is the same as the diagonal block c of A_i, to within scalar normalization and a transformation $A_i \to L_{i-1}^{-1} A_i L_i$.

Equations (2.4) simplify slightly if the model has only two-spin interactions between adjacent spins. Let $w_1(\sigma_i, \sigma_j), \ldots, w_6(\sigma_i, \sigma_p)$ be the Boltzmann weights of the edges $1, \ldots, 6$, as shows in Figs. 1 and 7. Then (sharing the edge weights between adjacent triangles)

$$w_i(a, b) = w_{i+3}(b, a), \qquad i = 1, 2, 3 \tag{2.21}$$

$$\Omega_i(a, b, c) = [w_{i-1}(a, b)\, w_{i+1}(b, c)\, w_{i+3}(c, a)]^{1/2} \tag{2.22}$$

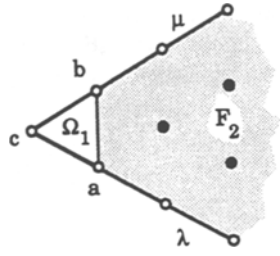

Fig. 6. The lattice segment corresponding to $(U_2^c)_{\lambda\mu}$, as defined by (2.13).

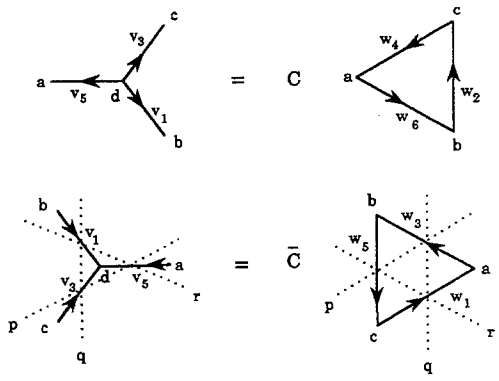

Fig. 7. Graphical representation of the star-triangle relation (2.26). The rapidities p, q, r are associated with the dotted lines.

Define matrices $G_1,\ldots, G_6$, $H_1,\ldots, H_6$ by

$$(G_i)_{\lambda\mu} = w_i(a, b)^{-1/2} F^{(i)}_{\lambda\mu}, \qquad (H_i)_{\lambda\mu} = w_i(a, b)^{1/2} F^{(i)}_{\lambda\mu} \tag{2.23}$$

where $a = E(\lambda)$ and $b = E(\mu)$. Then the equations can be written

$$(G_i A_{i+1} A_{i+2} H_{i+3})^* = (H_i A_{i+1} A_{i+2} G_{i+3})^* = \xi_i A_i A_{i+1} A_{i+2} A_{i+3} \tag{2.24}$$

$$H_{i-1} A_i H_{i+1} = \eta_i A_{i-1} G_i A_{i+1} \tag{2.25}$$

where $i = 1,\ldots, 6$, all indices are to be interpreted modulo 6, and we have the auxiliary condition

$$(H_i)_{\lambda\mu} = w_i(a, b)(G_i)_{\lambda\mu} \tag{2.26}$$

with $a = E(\lambda)$, $b = E(\mu)$.

2.3. *Z*-Invariance Properties

The above remarks apply to any triangular lattice edge-interaction model (and generalize trivially to any "interaction-round-a-triangle" model). Now let us consider models that satisfy the star-triangle relation shown in Fig. 7, i.e., there exist edge weight functions $v_1(a, b)$, $v_3(a, b)$, and $v_5(a, b)$ and coefficients C and $\bar{C}$ such that

$$\begin{aligned} \sum_d v_5(d, a)\, v_1(d, b)\, v_3(d, c) &= C w_2(b, c)\, w_4(c, a)\, w_6(a, b) \\ \sum_d v_5(a, d)\, v_1(b, d)\, v_3(c, d) &= \bar{C} w_5(b, c)\, w_1(c, a)\, w_3(a, b) \end{aligned} \tag{2.27}$$

Then the triangular lattice model is equivalent to a model on the honeycomb lattice, with edge interaction functions v_1, v_3, v_5. Further, all the nontrivial solutions that have been found (including the Ising and chiral Potts models) contain three arbitrary variables ("rapidities") such that w_1, w_4, v_1 depend on q and r, but not on p. Similarly, w_2, w_5, v_2 depend on r and p, but not on q; w_3, w_6, v_3 depend on p and q, but not on r. In fact there exist two generic functions $W_{p,q}(a, b)$ and $\bar{W}_{p,q}(a, b)$ and a mapping R of p ($p \to Rp$) such that

$$\bar{W}_{p,q}(a, b) = W_{q,Rp}(a, b), \qquad W_{p,q}(b, a) = \bar{W}_{q,Rp}(a, b) \tag{2.28}$$

$$\begin{aligned} w_1(a, b) &= W_{qr}(a, b), & w_2(a, b) &= \bar{W}_{pr}(a, b) \\ w_6(a, b) &= W_{pq}(a, b), & v_1(a, b) &= \bar{W}_{qr}(b, a) \\ v_5(a, b) &= W_{pr}(b, a), & v_3(a, b) &= \bar{W}_{pq}(a, b) \end{aligned} \tag{2.29}$$

$$W_{qp}(a, b) = 1/W_{pq}(a, b), \qquad \sum_c \bar{W}_{pq}(a, c)\, \bar{W}_{qp}(c, b) = S_{pq}\, \delta(a, b) \tag{2.30}$$

$$W_{pp}(a, b) = 1, \qquad \bar{W}_{pp}(a, b) = \delta(a, b) \tag{2.31}$$

where S_{pq} is some function of p and q.

The rapidities p, q, r are associated with the lines of the medial graph of the triangular lattice, which is a Kagomé lattice (Fig. 7 here; Fig. 12.10 of ref. 12). In general the rapidities can differ from line to line.

Exhibit the dependence of the matrices A_i and F_i on p, q, and r and consider an element of the matrix product $A_1(p, q, r)\, A_2(p, q, r')\, A_3(p', q, r')$. This is the partition function of a lattice lying in the right half-plane, with vertical rapidity q and other permitted rapidities p, p', r, r'. The spin values on the left-hand vertical edge depend on the element selected, and the ratio of two such elements is the ratio of the corresponding edge-spin correlations. The star-triangle relations (2.27) ensure that the model is "Z-invariant,"[48] which means that in the thermodynamic limit spin correlations depend only on the rapidities lying between the particular spins under consideration. The elements of the matrix product are therefore independent of q, so

$$A_1(p, q, r)\, A_2(p, q, r')\, A_3(p', q, r') = \text{independent of } q \tag{2.32}$$

[to within scalar factors that be removed by choosing an appropriate normalization: Eqs. (2.24)–(2.31) are independent of the normalization of the matrices A_i, G_i]. Fixing p' and r' and assuming that the matrices A_i are

invertible, it follows that there exist matrices C_{rp} and B_{pq}, functions or r, p and p, q, respectively, such that

$$A_1(p, q, r) = C_{rp} B_{pq} \tag{2.33}$$

Rotating the lattice anticlockwise through 60° is equivalent to replacing p, q, r by q, r, Rp, so

$$A_2(p, q, r) = C_{R_{p,q}} B_{qr} \tag{2.34}$$

and similarly for $A_3,..., A_6$. Substituting these forms back into (2.32), we find that we can choose B_{pq} so that

$$C_{R_{p,q}} = B_{pq}^{-1} \times (\text{constant matrix}) \tag{2.35}$$

When $p = r$, it follows from (2.30) that $A_2 = 1$ (the identity matrix), so the constant matrix in (2.35) is the identity and

$$A_2(p, q, r) = B_{p,q}^{-1} B_{qr}, \qquad B_{qp} = B_{pq} \tag{2.36}$$

(If we perturb about the case $r = p$, then to first order, $A_2 = 1 + \mathscr{B}_{pq}$, where $\mathscr{B}_{pq}$ is a generalization of the boost[49] operator $\mathscr{B}_p$ discussed in ref. 38.)

More generally, if we define $p_0,..., p_7$ as $R^{-1}r$, p, q, r, Rp, Rq, Rr, R^2p and write B_{pq} alternatively as $B(p, q)$, then

$$A_i(p, q, r) = B(p_{i-1}, p_i)^{-1} B(p_i, p_{i+1}) \tag{2.37}$$

for $i = 1,..., 6$. Note that we have to modify the "modulo 6" convention for the rapidities:

$$p_{i+6} = R^2 p_i$$

(In fact, $p_{i+3} = Rp_i$.) For consistency, the matrix function B_{pq} must satisfy the periodicity condition

$$B_{R^2p, R^2q} = M B_{pq} \tag{2.38}$$

where M is a constant matrix, independent of all rapidities. The matrices B_{pq}, M have the same block-diagonal structure as the corner transfer matrices A_i; in particular, $M^* = M$, and

$$A_1 \cdots A_6 = B_{R^{-1}r, p}^{-1} M B_{R^{-1}r, p} \tag{2.39}$$

We can always multiply B_{pq} on the left by an invertible constant matrix so as to diagonalize M: we then regain (1.1). We see that the M therein is the same as that defined by (2.38), and so is constant.

Similarly, the matrix product $A_1 F_2 A_3$ is independent of q, from which it follows that there is a matrix function F_{pr} such that

$$F_2(p, q, r) = B_{pq}^{-1} F_{pr} B_{qr} \tag{2.40}$$

satisfying the periodicity condition

$$F_{R^2 p, R^2 r} = M F_{pr} M^{-1} \tag{2.41}$$

There are corresponding function G_{pr}, H_{pr} for G_2, H_2, in particular,

$$H_2(p, q, r) = B_{pq}^{-1} H_{pr} B_{qr} \tag{2.42}$$

Substituting these forms back into (2.24)–(2.26), we find that the matrices B cancel out, leaving

$$(G_{pr} H_{Rp, Rr})^* = (H_{pr} G_{Rp, Rr})^* = \xi_{pr} 1 \tag{2.43}$$

$$H_{R^{-1}r, q} H_{q, Rq} = \eta_{pqr} G_{pr} \tag{2.44}$$

$$(H_{pr})_{\lambda\mu} = \bar{W}_{pr}(a, b)(G_{pr})_{\lambda\mu} = (\bar{W}_{pr}(a, b))^{1/2} (F_{pr})_{\lambda\mu} \tag{2.45}$$

where $\xi_2 = \xi_{pr}$ and $\eta_2 = \eta_{pqr}$. Since we have taken the thermodynamic limit of a large lattice, the matrices B_{pq}, F_{pr}, G_{pr}, H_{pr} are infinite dimensional and one has to be careful to use these equations only for values of p, q, r for which the matrix products exist and are associative. It appears that this is true when the Boltzmann weights $w_1, \ldots, v_5$ are positive and real. In terms of the variable u_p defined below, this means that u_p, u_q, u_r are real and

$$u_p < u_q < u_r < u_{Rp} \tag{2.46}$$

(if $u_p < u_q$, then $u_{Rp} < u_{Rq}$). The equations can be analytically continued away from this physical regime, but how far one can do this has to be determined in any particular case.

2.4. The Matrix Function X_p

From (2.454), fixing p and assuming the matrix function H_{pr} is invertible, we can deduce that H_{pr} is the product of a matrix function of p times a matrix function of r (to within a scalar factor). Also, from (2.31), $H_2(p, p, p) = 1$. Hence $H_{pp} = 1$ and we can normalize H_{pr} so that

$$H_{pr} = X_p^{-1} X_r \tag{2.47}$$

where X_p is some single-rapidity matrix function. It must satisfy the periodicity condition

$$X_{R^2 p} = L X_p M^{-1} \tag{2.48}$$

where L is a constant matrix. Substituting this form for H back into (2.44), we find that η is a function only of p and r, i.e., $\eta_{pqr} \equiv \eta_{pr}$, and

$$G_{pr} = (1/\eta_{pr}) X^{-1}_{R^{-1}r} X_{Rp} \tag{2.49}$$

Also, (2.6) simplifies to

$$Z^{(0)} = \text{Trace } M, \qquad \eta_{pr} Z_2^{(1)} = Z_i^{(2)} = \text{Trace } L, \qquad \forall i \tag{2.50}$$

Defining $\alpha = (\text{Trace } L)/(\text{Trace } M)$, it follows that

$$\eta_{Rp,Rr} = \eta_{pr}, \qquad \eta_{pr} \xi_{pr} = \alpha \tag{2.51}$$

and (2.43) reduces to

$$(X_p^{-1} X_{R^2 p})^* = \alpha \cdot 1 \tag{2.52}$$

The relation (2.45) now becomes

$$\eta_{pr} (X_p^{-1} X_r)_{\lambda\mu} = \bar{W}_{pr}(a, b)(X^{-1}_{R^{-1}r} X_{Rp})_{\lambda\mu} \tag{2.53}$$

and we can regard this a matrix functional relation. Taking $r = p$, we regain (2.52), with $\alpha = \eta_{pp}$.

Equation (2.53) is the main result of this section. It defines X_p to within a normalization and a constant equivalence transformation (block-diagonal on the right). Then M is given (to within a block-diagonal similarity transformation) by (2.48), so $\langle s(a) \rangle$ can be obtained from (2.13). The partition function per site κ is given by

$$\kappa = \eta_{pr} \eta_{q,Rp} \eta_{r,Rq} / \alpha \tag{2.54}$$

These observations on the p, q, r dependence of the matrices are true in the infinite-lattice limit for the original definitions of A_i, F_i. However, they are unaffected by the transformation $A_i \to L_{i-1}^{-1} A_i L_i$, provided L_i depends only on p_i and p_{i+1}. From (2.37), it is still possible to choose the L_i to diagonalize the A_i, so Eqs. (2.32)–(2.54) can be satisfied in a representation in which the A_i are diagonal.

2.5. The Matrices U, P, Q

After (2.19) we outlined an iterative procedure for solving the corner transfer matrix equations. This procedure simplifies somewhat if we use the Z-invariance properties, as we shall now show.

The matrices U_i, P_i, Q_i are also functions of the rapidities p, q, r. Since $A_1 F_2 A_3$ is independent of q, so are P_2 and Q_2, and we can write them as P_{pr} and Q_{pr}, respectively.

The matrix U_2^c is represented graphically in Fig. 6. Clearly it is a corner transfer matrix like A_2, but with one more column of spins, and the center spin is fixed to be c. It should therefore factor as A_2 does in (2.36), except that we are no longer free to normalize away an overall scalar factor. Hence

$$U_2^c(p, q, r) = \gamma_{pqr}(D_{pq}^c)^{-1} D_{qr}^c \tag{2.55}$$

where γ_{pqr} is a scalar and D_{pq}^c a matrix.

Like A_2, the matrix $U_2^c(p, q, r)$ is unchanged by a full rotation, i.e., by replacing p, q, r by R^2p, R^2q, R^3r. Writing D_{pq}^c alternatively as $D(p, q)^c$, there must therefore exist a constant matrix $\mathcal{M}^c$ such that, to within a possible scalar factor independent of c,

$$D(R^2p, R^2q)^c = \mathcal{M}^c D(p, q)^c \tag{2.56}$$

We can multiply D_{pq}^c on the left by a constant matrix so as to make $\mathcal{M}^c$ diagonal, with decreasing diagonal elements. (This is not always necessary; it may be sufficient to put $\mathcal{M}^c$ into a block-diagonal form, where each block has a certain magnitude.) We can also normalize the matrices so that the top left elements of D_{pq}^0 and $\mathcal{M}^0$ are unity: then there is no additional scalar factor in (2.56).

Remembering that U_{i+1} is obtained from U_i by replacing p, q, r by q, r, Rp, it follows that, to within a scalar factor independent of c,

$$U_1 \cdots U_6 = (D_{r'p}^c)^{-1} \mathcal{M}^c D_{r'p}^c \tag{2.57}$$

where for the remainder of this section we write $R^{-1}r$ as r'. Hence the eigenvalues of $U_1 \cdots U_6$ are the diagonal elements of $\mathcal{M}^0, \ldots, \mathcal{M}^{N-1}$: we select the largest n of all of these, m_c being from $\mathcal{M}^c$. Thus $n = m_0 + \cdots + m_{N-1}$. We form M as the n by n diagonal matrix with these elements. From (2.17) and (2.18), the columns of P_2^c are right eigenvectors of $U_1 \cdots U_6$, while the rows of Q_5^c are left eigenvectors. Let $|D_{pq}^c$ be the m_c by n matrix consisting of the first m_c rows of D_{pq}^c. Similarly, let $D_{pq}^{c^{-1}}|$ be the n by m_c matrix consisting of the first m_c columns of $D_{pq}^{c^{-1}}$. Then

$$P_{pr}^c = \pi_{pr} D_{r'p}^{c^{-1}}| \, B_{r,Rp}^c \tag{2.58}$$

where π_{pr} is some scalar factor and B_{pq}^c is a diagonal m_c by m_c matrix, as yet arbitrary.

We obtain a similar equation for Q_{pr}^c, involving another m_c by m_c diagonal matrix. This second matrix can be calculated from (2.19), giving

$$Q_{pr}^c = (\xi_{pr}/\pi_{Rp,Rr}) B_{r'p}^{c^{-1}} \, |D_{r,Rp}^c \tag{2.59}$$

We can fix π_{pr} and ξ_{pr} by requiring that the top left elements of B^0_{pq}, P^0_{pr}, Q^0_{pr} all be unity. We are also free to normalize the columns to P^c_{pr} in any convenient way consistent with rotation invariance [for the chiral Potts model we should also ensure that (2.85) is satisfied], i.e., $P(R^2p, R^2r)^c = P(p, r)^c$. The diagonal matrix B^c_{pq} is then determined by (2.58). On substituting (2.58) and (2.59) into (2.17) and (2.18), we find they are satisfied provided (2.51) holds and

$$\gamma_{pqr} = \eta_{q,Rp}\pi_{q,Rp}/\pi_{r,Rq} \tag{2.60}$$

$$A_2(p, q, r)^a = B^{a^{-1}}_{pq} B^a_{qr} \tag{2.61}$$

Thus the matrix B^a_{pq} that we have constructed is the block a of the matrix B_{pq} introduced in (2.36). We can form the matrix B_{pq} as the n by n diagonal matrix whose elements are the elements of $B^0_{pq},\ldots, B^{N-1}_{pq}$, arranged in the order that the corresponding elements of $\mathscr{M}^0_{pq},\ldots,\mathscr{M}^{N-1}_{pq}$ take in the matrix M. Then it satisfies the relation (2.38). Since γ_{pqr} is determined by (2.55), (2.60) gives $\eta_{q,Rp}$.

The final step in the iteration procedure after (2.19) is to form the matrices F_i, or equivalently F_{pr}. From (2.15), (2.16), and (2.40)

$$\begin{aligned} F_{pr} &= \pi_{pr} B_{r'p} [D^{-1}_{r'p}]_{\text{cols}} \\ &= (\xi_{pr}/\pi_{Rp,Rr})[D_{r,Rp}]_{\text{rows}} B^{-1}_{r,Rp} \end{aligned} \tag{2.62}$$

where $[D^{-1}_{pq}]_{\text{cols}}$ is the n by n matrix whose columns are the $n = m_0 + \cdots + m_{N-1}$ selected columns of $(D^0_{pq})^{-1},\ldots,(D^{N-1}_{pq})^{-1}$, arranged in the same order as the corresponding diagonal elements of M; similarly, $[D_{pq}]_{\text{rows}}$ is the matrix whose rows are the selected rows of $D^0_{pq},\ldots, D^{N-1}_{pq}$.

As in step (iv) of the iteration procedure, the upper right and diagonal elements of F_{pr} should be calculated from the first of Eq. (2.62), the lower left from the second. If n has increased since the last iteration, there will be come elements at the bottom right that are undetermined: these can be obtained by calculating G_{pr} from (2.44), (2.45).

Note that the matrix B_{pq} in (2.62) is the original n by matrix, not the new one calculated from (2.58) and (2.59). We should use the new one at the start of the next iteration, when calculating U_2 from (2.14) and (2.40).

2.6. Reflection Symmetry

The models under consideration also have an overall reflection symmetry. There exists a mapping S ($p \to Sp$) such that

$$W_{Sq,Sp}(a, b) = W_{pq}(a, b), \qquad \bar{W}_{Sq,Sp}(a, b) = \bar{W}_{pq}(b, a) \tag{2.63}$$

$$S^2 = (RS)^2 = 1 \tag{2.64}$$

Replacing p, q, r by Sr, Sq, Sp leaves (2.46) unchanged and is equivalent to a vertical reflection of the entire lattice. In particular,

$$A_1(Sr, Sq, Sp) = A_3(p, q, r)^T, \qquad A_2(Sr, Sq, Sp) = A_2(p, q, r)^T \tag{2.65}$$

the T denoting matrix transposition. Corresponding equations hold with A replaced by F, G, or H.

Using (2.36), (2.38)–(2.41) (with their rotated analogs), (2.47), and (2.48), it follows that there exist symmetric constant matrices Y, Γ such that

$$B_{Sq,Sp} = Y^{-1}(B_{pq}^T)^{-1}, \qquad F_{Sr,Sp} = Y^{-1}F_{pr}^T Y \tag{2.66}$$

$$G_{Sr,Sp} = Y^{-1}G_{pr}^T Y, \qquad H_{Sr,Sp} = Y^{-1}H_{pr}^T Y \tag{2.67}$$

$$X_{Sp} = \Gamma^{-1}(X_p^T)^{-1} Y \tag{2.68}$$

$$YM = (YM)^T = M^T Y, \qquad \Gamma L = (\Gamma L)^T = L^T \Gamma \tag{2.69}$$

$$\eta_{Sr,Sp} = \eta_{pr} \tag{2.70}$$

The matrices M, Y are block-diagonal.

2.7. Inversion Relations for η_{pq}

The models we have in mind also have the property that

$$\pi_{pq} = \prod_{b=1}^{N} W_{pq}(a, b) = \prod_{b=1}^{N} W_{pq}(b, a) = \text{independent of } a \tag{2.71}$$

Define

$$f_{pq} = \{[\det_N \bar{W}_{pq}(a, b)]/\pi_{pq}\}^{1/N} \tag{2.72}$$

where $\det_N \bar{W}_{pq}(a, b)$ is the determinant of the N by N matrix with entries $\bar{W}_{pq}(a, b)$ in positions (a, b). If we fix a in the star-triangle relations (2.27), we can write each side as the element (b, c) of a matrix product. Taking determinants, we get

$$C = \bar{C} = f_{pq} f_{qr}/f_{pr} \tag{2.73}$$

in agreement with the conjecture of ref. 8 and the proof of ref. 10. If instead we fix b or c, we find that

$$C = \bar{C} = f_{pq} f_{r,Rp}/f_{r,Rq} = f_{r,Rp} f_{q,r}/f_{q,Rp} \tag{2.74}$$

and hence

$$f_{pq} f_{q,Rp} = f_{qr} f_{r,Rq} = f_{pr} f_{r,Rp} \tag{2.75}$$

The common value of these expressions is therefore independent of all of p, q, and r, i.e., is a rapidity-independent constant. [For the chiral Potts model it is $N/(k')^{(N-1)/N}$: Eq. (3.35) of ref. 13.]

Comparing (2.54) with (2.11) of ref. 13, the ψ_{pq}, $\bar{\psi}_{pq}$ therein are related to our η_{pq} by

$$e^{-\psi_{pq}} = \frac{\lambda_q \eta_{q,Rp}}{\gamma \lambda_p}, \qquad e^{-\bar{\psi}_{pq}} = \frac{\gamma^2 \lambda_p \eta_{pq}}{\alpha \lambda_q} \tag{2.76}$$

where γ is a constant and λ_p some single-rapidity function. Also, taking $r = Rp$ in (2.45) and using (2.31), we obtain $\eta_{p,Rp} = 1$. From (2.13) and (3.40) of ref. 13, it follows that $\gamma = 1$ and

$$\eta_{q,Rp} = h_q \eta_{pq} / [\alpha h_p f_{pq}] \tag{2.77}$$

$$\eta_{pq}\eta_{qp} = \alpha^2 f_{pq} f_{qp}, \qquad \eta_{pq}\eta_{R^{-1}q,Rp} = 1 \tag{2.78}$$

where h_p is some other single-rapidity function. [The last of these equations follows easily from (2.45), (2.28), and (2.30).]

These last two equations are the "inversion relations." They relate η_{pq} to its values under the mappins $p, q \to q, p$ and $p, q \to R^{-1}q, Rp$. These have fixed points at $q = p$ and $q = Rp$, which are the endpoints of the physical regime. Some models have the "difference property" that W_{pq} and $\bar{W}_{pq}$ are functions only of $p - q$. For these (with appropriate normalizations of W_{pq} and $\bar{W}_{pq}$), η_{pq} is analytic in some domain containing the points $q = p$ and $q = Rp$. The inversion relations then define η_{pq} and provide a convenient way to calculate it.[50] So far, however, this program has not been carried out for the chiral Potts model, which does not have the difference property and whose analyticity properties are much more complicated. We do have expressions for η_{pq},[13,21,22] but they are quite cumbersome. One of the aims of this paper is to work toward a more transparent formulation of the solution of the chiral Potts model.

2.8. Square Lattice

We can apply our results also to the honeycomb lattice (via the star-triangle relation), or to the square lattice (by specialization). For the latter case, take $r = q$. Then $w_1(a, b) = w_4(b, a) = W_{pp}(a, b) = 1$, so there is no interaction in the SW–NE direction. The remaining edges of $\mathcal{L}$ form a parallelogram lattice which can be distorted to a square lattice. The interactions in the two direction are $W_{pq}(a, b)$ and $\bar{W}_{pq}(a, b)$.

The same result can be achieved by taking $q = p$, then replacing r by q, but the lattice segments corresponding to the CTMs $A_1, \ldots, A_6$ are

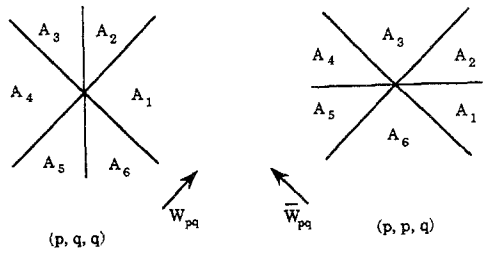

Fig. 8. Two ways of obtaining the square lattice from the triangular: replacing (p, q, r) by (p, q, q) or by (p, p, q).

different: they are shown in Fig. 8. Some correspond to 45° segments, others to 90° quadrants.

The 90° CTMs $\tilde{A}_{pq},\ldots,\tilde{D}_{pq}$ of the square lattice were discussed in ref. 38. We see that they can be expressed in terms of our triangular lattice matrices $A_i(p, q, r)$:

$$\begin{aligned} \tilde{A}_{pq} &= A_6(p, q, q)\, A_1(p, p, q), \qquad \tilde{B}_{pq} = A_2(p, p, q)\, A_2(p, q, q) \\ \tilde{C}_{pq} &= A_3(p, q, q)\, A_4(p, p, q), \qquad \tilde{D}_{pq} = A_5(p, p, q)\, A_5(p, q, q) \end{aligned} \tag{2.79}$$

Using (2.37) and (2.38), it follows (after cancellations) that

$$\begin{aligned} \tilde{A}_{pq} &= B^{-1}_{R^{-1}q,\, R^{-1}q} B_{pp}, \qquad \tilde{B}_{pq} = B^{-1}_{pp} B_{qq} \\ \tilde{C}_{pq} &= B^{-1}_{qq} B_{Rp,Rp}, \qquad \tilde{D}_{pq} = B^{-1}_{Rp,Rp} B_{Rq,Rq} \end{aligned} \tag{2.80}$$

These are precisely the relations (14) of ref. 38, the matrix function A_p therein being our B_{pp}. The matrices M and Y^{-1} therein are the same as our M and Y.

Note that $\tilde{A}_{pq},\ldots,\tilde{D}_{pq}$ are the CTMs of the square lattice drawn diagonally. The CTMs of the usual square lattice can also be obtained from Fig. 8: they are $A_1(p, q, q)$, $A_3(p, p, q)$, $A_4(p, q, q)$, and $A_6(p, p, q)$. In every case the product of the CTMs is M (to within a block-diagonal similarity transformation) and $\langle s(a) \rangle$ is given by (2.13). For the square lattice one can also define matrices F_i, G_i, H_i and write down equations analogous to (2.24), (2.25). One reason why we have chosen to work with the triangular lattice is that (2.25) is only quadratic in the F_i, G_i, H_i, whereas for the square lattice its analog is cubic [Eqs. (30c), (30d) of ref. 39].

2.9. Chiral Potts Model

From now on we focus attention on the case of the chiral Potts model (of which the Ising model is a special case). This is formulated[8,9] in terms

of homogeneous sets of variables (a, b, c, d) satisfying the relations (only two of which are independent)

$$\begin{aligned} a^N + k'b^N &= kd^N, \qquad k'a^N + b^N = kc^N \\ ka^N + k'c^N &= d^N, \qquad kb^N + k'd^N = c^N \end{aligned} \tag{2.81}$$

Here N is the number of states of each spin and k, k' are real constants satisfying $k^2 + k'^2 = 1$.

There is one such set of variables a_p, b_p, c_p, d_p for each rapidity p. The Boltzmann weights are given by $W_{pq}(a, b) = W_{pq}(a - b)$ and $\bar{W}_{pq}(a, b) = \bar{W}_{pq}(a - b)$, where $W_{pq}(n) \equiv W_{pq}(n + N)$, $\bar{W}_{pq}(n) \equiv \bar{W}_{pq}(n + N)$, and

$$W_{pq}(n) = \prod_{j=1}^{n} \frac{d_p b_q - a_p c_q \omega^j}{b_p d_q - c_p a_q \omega^j}, \qquad \bar{W}_{pq}(n) = \prod_{j=1}^{n} \frac{\omega a_p d_q - d_p a_q \omega^j}{c_p b_q - b_p c_q \omega_j} \tag{2.82}$$

where $\omega = \exp(2\pi i/N)$. The mapping R, S are specified by

$$a_{Rp}, b_{Rp}, c_{Rp}, d_{Rp} = b_p, \omega a_p, dp, c_p \tag{2.83}$$

$$a_{Sp}, b_{Sp}, c_{Sp}, d_{Sp} = \omega^{-1/2} c_p, d_{p,a}{}^{p}, \omega^{-1/2} b_p \tag{2.84}$$

In Eq. (48) of ref. 43 it was shown that there is a uniformizing parametrization of the relations (2.81) in terms of hyperelliptic functions.

In addition to the rotation and reflection symmetries discussed in Section 2, this model has a further symmetry: rotating the lattice through 180° is equivalent to both replacing the rapidities p, q, r by Rp, Rq, Rr, and to replacing each spin a by its negative $-a$ (mod N). [The first equivalence follows from (2.28), the second follows from Z_N invariance and is consistent with our having chosen the boundary spins to be zero.] Hence in the original spin representation

$$\begin{aligned} A_2(Rp, Rq, Rr) &= \mathscr{P}^{-1} A_2(p, q, r) \mathscr{P} \\ F_2(Rp, Rq, Rr) &= \mathscr{P}^{-1} F_2(p, q, r) \mathscr{P} \end{aligned} \tag{2.85}$$

where $\mathscr{P}$ is a permutation matrix that replaces a spin state $\{\lambda_1, \lambda_2, \ldots\}$ by $\{-\lambda_1, -\lambda_2, \ldots\}$. Hence $\mathscr{P} = \mathscr{P}^{-1} = \mathscr{P}^T$.

From (2.36), (2.40), and (2.47), it follows that there exist invertible constant matrices $\tilde{L}$, $\tilde{M}$ (with $\tilde{M}$ block-diagonal) such that

$$\begin{aligned} B_{Rp,Rq} &= \tilde{M} B_{pq} \mathscr{P} \\ F_{Rp,Rr} &= \tilde{M} F_{pr} \tilde{M}^{-1} \\ X_{Rp} &= \tilde{L} X_p \tilde{M}^{-1} \end{aligned} \tag{2.86}$$

and similarly with F replaced by G or H. These relations imply the ful rotation symmetries (2.38), (2.41), (2.48), with

$$L=\tilde{L}^2, \qquad M=\tilde{M}^2 \tag{2.87}$$

and $\Gamma\tilde{L}$, $Y\tilde{M}$, $Y\tilde{M}^{-1}$ are symmetric.

Equations (2.43)–(2.53) are unchanged by the transformation $G_{pr}\to DG_{pr}D^{-1}$, $H_{pr}\to DH_{pr}D^{-1}$, $X_p\to KX_pD^{-1}$, where D, K are any invertible constant matrices. This transformation takes $\tilde{L}$, $\tilde{M}$, Γ, Y, $Y\tilde{M}^{-1}$ to $K^{-1}\tilde{L}K$, $D^{-1}\tilde{M}D$, $K^T\Gamma K$, D^TYD, $D^TY\tilde{M}^{-1}D$. It appears (at least for the chiral Potts model) that Γ and $Y\tilde{M}^{-1}$ are positive definite, so we can choose K and D so that $\Gamma=Y\tilde{M}^{-1}=1$. Then $\tilde{L}$ and $\tilde{M}$ become symmetric, and we can choose D so as to diagonalize M. We use such a representation in Section 4. (We can if we wish further choose K so as to make L diagonal.)

3. ISING MODEL

The simplest nontrivial case of the chiral Potts model is when $N=2$, when we regain the Ising model. If ρ_1, s_1, $\tau_{1,1}$ are the variables of ref. 43 and we write them simply as ρ, s, τ, then $\tau=2\rho$ and the parametrization (48) therein reduces to

$$a_p : b_p : c_p : d_p = i\theta_1(i\tilde{u}_p) : -\theta_4(i\tilde{u}_p) : \theta_3(i\tilde{u}_p) : \theta_2(i\tilde{u}_p) \tag{3.1}$$

where $\tilde{u}_p=-i\pi[s+(1-\tau)/4]$ and $\theta_1(u),\ldots,\theta_4(u)$ are the usual Jacobi theta functions of argument u and nome $x=e^{i\pi\tau}$ (§8.181 of ref. 51). With the notation of ref. 43 and (2.81), these theta functions ar of modulus k' rather than k, so that

$$k'=4x^{1/2}\prod_{n=1}^{\infty}\left(\frac{1+x^{2n}}{1+x^{2n-1}}\right)^4, \qquad \tau=\frac{iK}{K'} \tag{3.2}$$

K, K' are the usual elliptic integrals of modulus k.

We shall later consider the low-temperature limit, when x is small. Defining

$$\zeta_p=-ix\exp(-2\tilde{u}_p)=\exp[i\pi(s+\tau/2)] \tag{3.3}$$

if ζ_p is of order unity; then to relative first order in a expansion in powers of x,

$$\begin{aligned} a_p&=e^{i\pi/4}\zeta^{1/2}x^{-1/4}(1+ix/\zeta_p)(1-ix\zeta_p), \qquad b_p=-(1-i\zeta_p)\\ c_p&=1+i\zeta_p, \qquad d_p=e^{i\pi/4}\zeta^{1/2}x^{-1/4}(1-ix/\zeta_p)(1+ix\zeta_p) \end{aligned} \tag{3.4}$$

(to within an overall normalization factor).

Returning to the general case $0<x<1$, define

$$u_p = 2K'\tilde{u}_p/\pi, \qquad u_q = 2K'\tilde{u}_q/\pi \tag{3.5}$$

Then from (2.82) it follows that

$$W_{pq}(0) = 1, \qquad W_{pq}(1) = \frac{\operatorname{cn} u_q + \operatorname{sn} u_p \operatorname{dn} u_q}{\operatorname{cn} u_p + \operatorname{dn} u_p \operatorname{sn} u_q} \tag{3.6}$$

$$\bar{W}_{pq}(0) = 1, \qquad \bar{W}_{pq}(1) = \frac{k'(\operatorname{sn} u_q - \operatorname{sn} u_p)}{\operatorname{dn} u_p \operatorname{cn} u_q + \operatorname{dn} u_q \operatorname{cn} u_p} \tag{3.7}$$

where sn, cn, and dn are the Jacobi elliptic functions of modulus k.

These formulas simplify to

$$W_{pq}(1) = k' \operatorname{scd}(u_p - u_q + K), \qquad \bar{W}_{pq}(1) = k' \operatorname{scd}(u_q - u_p) \tag{3.8}$$

where $\operatorname{scd}(u)$ is the function $\operatorname{sn}(u/2)/[\operatorname{cn}(u/2)\ \operatorname{dn}(u/2)]$. Also, if J and $\bar{J}$ are the usual dimensional Ising model interaction coefficients (H and H' in ref. 52), then $W_{pq}(1) = \exp(-2J)$, $\bar{W}_{pq}(1) = \exp(-2\bar{J})$. Using (2.72) and setting $u = u_p - u_p$, we find

$$\sinh 2J = \frac{\operatorname{sn} u}{\operatorname{cn} u}, \qquad \sinh 2\bar{J} = \frac{\operatorname{cn} u}{k' \operatorname{sn} u} \tag{3.9}$$

$$f_{pq} = \frac{H_1(0)\,\Theta_1(0)\,H_1((K-u)/2)\,\Theta_1((K-u)/2)}{H_1(K/2)\,\Theta_1(K/2)\,H_1(u/2)\,\Theta_1(u/2)}, \qquad k' = \frac{1}{\sinh 2J \sinh 2\bar{J}} \tag{3.10}$$

where $H_1(u)$ and $\Theta_1(u)$ are the usual Jacobi theta functions of modulus k. For $0<k<1$ the model is in the ordered ferromagnetic phase.

Note that $W_{pq}(n)$ and $\bar{W}_{pq}(n)$ depend on p and q only via the difference $u_q - u_p$. This is the "difference property" mentioned above.

From Eqs. (37) and (39) of ref. 43, the mapping R and S take s to $s+\tau/2$ and $-s-\tau$, respectively, so

$$u_{Rp} = u_p + K, \qquad u_{Sp} = -3K - iK' - u_p \tag{3.11}$$

and $\zeta_{Rp} = x\zeta_p$, $_{Sp} = 1/(x\zeta_p)$.

As yet we have not defined the rapidity variables p, q, r precisely, having used them really only as labels: for the rest of this section let us take

$$p = u_p + (K + iK')/2 \tag{3.12}$$

Then in terms of the variable $s = s_1$ of ref. 43, $p = -2iK's$.

3.1. Partition Function per Site

We can use the inversion identities (2.78) to calculate η_{pq}. First we use the conjugate modulus expansions of the elliptic functions. Define, consistently with (3.2),

$$z = e^{-\pi(q-p)/K'}, \qquad x = e^{-\pi K/K'} \tag{3.13}$$

$$\begin{aligned} B(z) &= \prod_{n=1}^{\infty} (1-x^{4n-2}z)(1-x^{4n-2}/z) \\ C(z) &= \prod_{n=1}^{\infty} (1-x^{2n}/z)^n/(1-x^{2n}z)^n \end{aligned} \tag{3.14}$$

Then

$$\begin{aligned} f_{pq} &= z^{-1/4}B(1)\, B(x/z)/[B(x)\, B(z)] \\ W_{pq}(1) &= z^{1/2}B(xz)/B(x/z) \\ \bar{W}_{pq}(1) &= q^{1/2}z^{-1/2}B(x^2/z)/B(z) \end{aligned} \tag{3.15}$$

$$C(z)\, C(1/z) = 1, \quad C(z)\, C(q^2/z) = B(z)\, B(q^2/z), \quad C(q) = B(q) \tag{3.16}$$

The function η_{pq}, like f_{pq}, W_{pq}, and $\bar{W}_{pq}$, can depend on p and q only via the difference $u_q - u_p$, i.e., via z. Writing f_{pq} and η_{pq} as $f(z)$ and $\eta(z)$, we find that the relations (2.78) become

$$\eta(z)\, \eta(z^{-1})/\eta(1)^2 = f(z)\, f(z^{-1}), \qquad \eta(z)\, \eta(q^2/z) = 1 \tag{3.17}$$

We make the standard assumption[53,50] that $\ln \eta(z)$ is single-valued and analytic in the annulus $q \leqslant |z| \leqslant 1$, except possibly for singularities due to poles or zeros of $\eta(z)$ at the inversion points $z = 1$ and $z = q$. Then the relations (3.17) define $\eta(z)$ uniquely. It is a meromorphic function with poles or zeros only at $z = 1, q^{\pm 1}, q^{\pm 2}, \ldots$:

$$\eta(z) = C(z)\, C(q/z)/B(z) \tag{3.18}$$

in agreement with past results.[52,54] The relation (2.77) is satisfied with $h_p = \exp(\pi u_p/4K')$.

Define, for $j = 1$ or 3,

$$\phi_j(z) = \prod_{n=1}^{\infty} \left(\frac{1-x^{2n-1}z}{1-x^{2n}z}\right)^n \frac{1}{1-x^{4n-j}z} \tag{3.19}$$

Then

$$W_{pq}(1) = \frac{z^{1/2}\phi_3(z)\,\phi_1(z^{-1})}{\phi_3(z^{-1})\,\phi_1(z)} \tag{3.20}$$

$$\eta_{q-K,p} = \eta(x/z) = \phi_3(z)/\phi_3(z^{-1}) \tag{3.21}$$

3.2. Elements of the CTMs

As is discussed in ref. 38 and in §13.5 of ref. 12, the difference property enables us to calculate the eigenvalues of the CTMs A_i and of their product M, and hence to obtain the spontaneous magnetization.

First note that $A_2(p, q, r)$ is a function only of $q-p$ and $r-q$. From (2.36), first fixing q and them grouping p-dependent terms on one side and r-dependent terms on the other, we find that the matrix B_{qr} must be of the form

$$B_{qr} = S_q \hat{B}_{r-q} \tag{3.22}$$

where $S_p^{-1}S_q$ is a function only of $q-p$. Choosing (as we can) $S_0 = 1$ and using the case $p=0$, we must have $S_p^{-1}S_q = S_{q-p}$, i.e.,

$$S_p S_r = S_{p+r} \qquad \forall p, r \tag{3.23}$$

Interchanging p and r, it follows that S_p, S_r commute, and so can be simultaneously diagonalized. (The required similarity transformation can be absorbed into the definitions of $\hat{B}_{r-q}$ and B_{qr} without affecting A_2.) Then (3.23) becomes a scalar equation, one for each eigenvalue, and by logarithmic differentiation it is easily seen that each eigenvalue of S_p must be an exponential in p, so in general

$$S_p = e^{-p\mathscr{B}} \tag{3.24}$$

where $\mathscr{B}$ is a constant diagonal matrix. So, using (2.38), we obtain

$$B_{pq} = e^{-p\mathscr{B}}\hat{B}_{q-p}, \qquad M = e^{-2K\mathscr{B}} \tag{3.25}$$

Similarly, the matrix H_2 is a function only of the differences $q-p$ and $r-q$, so from (2.42), H_{pr} is of the form

$$H_{pr} = e^{-p\mathscr{B}}\hat{H}_{r-p}e^{r\mathscr{B}} \tag{3.26}$$

Using (2.66), (3.11), (3.25), and the difference property, we can deduce that

$$\hat{B}_{q-p}\hat{B}^T_{q-p} = Y^{-1}e^{-2K\mathscr{B}}e^{(p-q)\mathscr{B}} \tag{3.27}$$

The matrix $D^2 = Y^{-1}e^{-2K\mathscr{B}}$ is symmetric and commutes with $\mathscr{B}$, and so can be chosen diagonal. In fact we can absorb it into B_{pq} and H_{pr} (by replacing them by DB_{pq} and $DH_{pr}D^{-1}$) without changing the A_i or H_i. Hence we can take $Y = e^{-2K\mathscr{B}}$. Then from (2.67) the matrix $\hat{H}_{r-p}$ is symmetric, and from the factorization property (2.47) we must have

$$\hat{H}_{r-p} = V^T e^{(p-r)\mathscr{C}} V \tag{3.28}$$

where C, V are constant matrices; C is diagonal and V is orthogonal. From (2.47), we can choose

$$X_p = e^{-p\mathscr{C}} V e^{p\mathscr{B}} \tag{3.29}$$

Substituting this form into (2.53), using (2.28), and changing variables, we get

$$W_{pq}(a, b)(V^T e^{(p-q-K)\mathscr{C}} V)_{\lambda\mu} = \eta_{q-K,p}(e^{(p-q)\mathscr{B}} V^T e^{(q-p-K)\mathscr{C}} V e^{(p-q)\mathscr{B}})_{\lambda\mu} \tag{3.30}$$

where as usual $a = E(\lambda)$, $b = E(\mu)$. Also, (2.52) gives

$$(V^T r^{-2K\mathscr{C}} V)^* = \alpha e^{-2K\mathscr{B}} = \alpha M \tag{3.31}$$

From (2.68), (2.86), and (2.87),

$$\begin{gathered} \mathscr{P} = 1, \qquad \tilde{M} = e^{-K\mathscr{B}}, \qquad Y = M = e^{-2K\mathscr{B}} \\ \tilde{L} = e^{-K\mathscr{C}}, \qquad \Gamma = L = e^{-2K\mathscr{C}} \end{gathered} \tag{3.32}$$

3.3. Calculation of $\mathscr{B}$ and $\mathscr{C}$

In a low-temperature expansion, we hold z fixed and expand in increasing powers of x. From (3), to any order, the coefficients of f_{pq}, $W_{pq}(1)$ and $\bar{W}_{pq}(1)$, are Laurent polynomials in $z^{1/4}$ (in fact they are $z^{n/4}$ multiplied by a Laurent polynomial in z, where n is some integer). They are periodic functions of p and q, of period $8iK'$, single-valued, and analytic in an infinite vertical strip. The same general property appears to be true of all the elements of all the various matrices, in particular of the diagonal matrices $e^{-p\mathscr{B}}$ and $e^{-p\mathscr{C}}$. Hence each of the diagonal elements of $\mathscr{B}$ and $\mathscr{C}$ must be of the form $\pi n/4K'$, where n is some integer (different for different elements).

We can calculate these integers. First note from (2.36), (2.42), (3.25), (3.26), and (3.28) that

$$\begin{aligned} A_2 &= \hat{B}^{-1}_{q-p} e^{(p-q)\mathscr{B}} \hat{B}_{r-q} \\ H_2 &= \hat{B}^{-1}_{q-p} V^T e^{(p-r)\mathscr{C}} V e^{(r-q)\mathscr{B}} \hat{B}_{r-q} \end{aligned} \tag{3.33}$$

In the low-temperature limit $x \to 0$, $u_p < u_q < y_r < u_p + K$, it appears that the matrices A_i, H_i, $\hat{B}_{q-p}$ are "near-diagonal," in the sense that their eigenvalues are their diagonal elements and that $V \to 1$. In particular, the diagonal elements of A_2 and H_2 can be obtained from Fig. 2 by taking all the spins in each vertical line to be equal.

For $i = 1, 2, \ldots$, let

$$\psi_{i,i+1} = \psi(\lambda_i, \lambda_{i+1}) = \lambda_{i+1} - \lambda_i \pmod 2 \tag{3.34}$$

so that $\psi_{i,i+1}$ is either 0 or 1. With the spin set $\lambda = \{\lambda_1, \lambda_2, \ldots\}$ associate two integers

$$\begin{aligned} [\lambda] &= \psi(\lambda_1, \lambda_2) + 3\psi(\lambda_2, \lambda_3) + 5\psi(\lambda_3, \lambda_4) + \cdots \\ \{\lambda\} &= \psi(\lambda_1, \lambda_2) + 2\psi(\lambda_2, \lambda_3) + 3\psi(\lambda_3, \lambda_4) + \cdots \end{aligned} \tag{3.35}$$

(We are considering the infinite lattice embedded in a sea of spins with value 0, so $\lambda_i \to 0$ as $i \to \infty$.) Then to leading order

$$\begin{aligned} (A_2)_{\lambda\lambda} &= [(w_1(1)\, w_6(1)]^{[\lambda]/2} \\ (A_2)_{\lambda\lambda}(H_2)_{\lambda\lambda} &= [(w_1(1)\, w_6(1)]^{2\{\lambda\}} \end{aligned} \tag{3.36}$$

where

$$w_1(1)\, w_6(1) = W_{qr}(1)\, W_{pq}(1) = e^{\pi(p-r)/2K'}$$

Taking A_2, H_2 to be diagonal matrices with elements given by (3.36), $\hat{B}_{q-p}$ to be diagonal, and $V = 1$, we obtain from (3.33) that

$$\hat{B}_{r-p} = A_2 = e^{(p-r)\mathscr{B}/2}, \qquad A_2 H_2 = e^{(p-r)\mathscr{C}} \tag{3.37}$$

Comparing this result with (3.36), we find

$$\mathscr{B}_{\lambda\lambda} = \pi[\lambda]/2K', \qquad \mathscr{C}_{\lambda\lambda} = \pi\{\lambda\}/K' \tag{3.38}$$

These results are true to leading order of x small. However, these last expressions are already of the form $\pi n/4K'$ (n an integer). We expect $\mathscr{B}_{\lambda\lambda}$ and $\mathscr{C}_{\lambda\lambda}$ to be continuous functions of x at least for $0 < x < 1$, i.e., $0 < k < 1$, which is the low-temperature phase of the Ising model. This implies that (3.38) is *exact* throughout this range: we have calculated the diagonal matrices $\mathscr{B}$ and $\mathscr{C}$.

We can regard $\psi_{12}, \psi_{23}, \psi_{34}$, etc., as independent variables, with values 0 or 1, and $\lambda_1, \lambda_2, \lambda_3, \ldots$, as defined by (3.34). Because of the additive

form of $[\lambda]$ and $\{\lambda\}$, $\mathscr{B}$ and $\mathscr{C}$ are then simple direct sums. Their exponentials are direct products of simple two-by-two diagonal matrices:

$$e^{2(p-q)\mathscr{B}} = \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & z^3 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & z^5 \end{pmatrix} \otimes \cdots \tag{3.39}$$

$$e^{(p-q)\mathscr{C}} = \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & z^2 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & z^3 \end{pmatrix} \otimes \cdots \tag{3.40}$$

The spontaneous magnetization follows in the usual way. Taking $s(a) = (-1)^a$ in (2.13) and noting that

$$(-1)^a = (-1)^{\lambda_1} = (-1)^{\psi_{12} + \psi_{23} + \psi_{34} + \cdots} \tag{3.41}$$

we obtain for the matrix S

$$S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \cdots \tag{3.42}$$

From (3.25), M is given by (3.39) with z replaced by x. It follows immediately that

$$\begin{aligned} \langle (-1)^a \rangle &= \frac{(1-x)(1-x^3)(1-x^5)\cdots}{(1+x)(1+x^3)(1+x^5)\cdots} \\ &= k^{1/4} \end{aligned} \tag{3.43}$$

This is the famous result of Onsager[55] and Yang.[56]

3.4. Calculation of V

We have obtained η_{pq}, $\mathscr{B}$, $\mathscr{C}$ from special properties, but in principle they are defined, along with the constant orthogonal matrix V, by (3.30). The matrix V is needed if we want to calculate correlations of adjacent spins; for instance, from Fig. 5c, using (2.22) and (2.23),

$$\langle s_1(a)\, s_2(b)\, s_3(c) \rangle = \frac{\text{Trace } S_1 A_1 H_2 S_2 A_3 H_4 S_3 A_5 H_6}{\text{Trace } A_1 H_2 A_3 H_4 A_5 H_6} \tag{3.44}$$

where $s_1(a)$, $s_2(a)$, $s_3(a)$ are arbitrary functions and S_1, S_2, S_3 are the corresponding diagonal matrices, defined analogously to (2.11). From (2.37) and (2.42), the numerator on the rhs of (3.44) is

$$\text{Trace } M S_1 H_{pr} S_2 H_{r,Rq} S_3 H_{Rq,R^2p}$$

Using (3.26), this becomes

$$\text{Trace } S_1 \hat{H}_{r-p} S_2 \hat{H}_{q-r+K} S_3 \hat{H}_{p-q+K}$$

Now using (3.28) and defining

$$\tilde{S}_i = V S_i V^T$$

we find that

$$\langle s_1(a)\, s_2(b)\, s_3(c)\rangle = \frac{\text{Trace } \tilde{S}_1 e^{(p-r)\mathscr{C}} \tilde{S}_2 e^{(r-q-K)\mathscr{C}} \tilde{S}_3 e^{(q-p-K)\mathscr{C}}}{\text{Trace } e^{-2K\mathscr{C}}} \tag{3.45}$$

(Note that we do not need $\hat{B}_{q-p}$: it cancels out of all the equations.)

We can calculate some of the elements of V directly from (3.30). Take $|z|<1$, and order the rows and columns of the matrices so that the diagonal elements of $\exp[2(p-q)\mathscr{B}]$ are nonincreasing: from (3.39) they are $1, z, z^3, z^4, z^5,\ldots$. The corresponding elements of $\exp[(p-q)\mathscr{C}]$ are $1, z, z^2, z^3, z^2,\ldots$, and those of S are $1, -1, -1, 1, -1,\ldots$. In terms of the spin state $\lambda = \{\lambda_1, \lambda_1,\ldots\}$, this means that the first five states we are considering are $\{000\ldots\}$, $\{1000\ldots\}$, $\{1100\ldots\}$, $\{0100\ldots\}$, $\{1110\ldots\}$; the corresponding values of $[\lambda]$ are given by (3.35) to be 0, 1, 3, 4, 5. With this ordering, label the rows and columns of the matrices simply as 1, 2, 3, 4, 5,..., and define

$$\begin{aligned} h_{ij}(z) &= (\hat{H}_{q-p+K})_{ij} \\ &= (V^T e^{(p-q-K)\mathscr{C}} V)_{ij} \\ &= V_{1i}V_{1j} + V_{2i}V_{2j}xz + V_{3i}V_{3j}x^2z^2 + (V_{4i}V_{4j} + V_{5i}V_{5j})x^3z^3 + \cdots \end{aligned} \tag{3.46}$$

We expect the matrix products in (3.30) to exist, at least for $|q-p|<K$, and hence expect the sum in (3.46) to converge for $|z|<1/x$. Thus $h_{ij}(z)$ should be an analytic function inside this circle.

With S defined by (3.42), let $\theta_{ij}=0$ if $S_{ii}=S_{jj}$, else $\theta_{ij}=1$. Then (3.30) can be written

$$W_{pq}(\theta_{ij})\, h_{ij}(z) = \eta_{q-K,p}\, h_{ij}(z^{-1}) \exp[(p-q)(\mathscr{B}_{ii}+\mathscr{B}_{jj})] \tag{3.47}$$

Define $\bar{h}_{ij}(z)$ so that

$$\begin{aligned} h_{ij}(z) &= \bar{h}_{ij}(z)\, \phi_3(z) \qquad \text{if} \quad S_{ii}=S_{jj} \\ &= \bar{h}_{ij}(z)\, \phi_1(z) \qquad \text{if} \quad S_{ii}\neq S_{jj} \end{aligned} \tag{3.48}$$

and set

$$n_{ij} = (K'/\pi)(\mathscr{B}_{ii} + \mathscr{B}_{jj}) - \tfrac{1}{2}\theta_{ij} \tag{3.49}$$

Then n_{ij} is a nonnegative integer and, using (3.20) and (3.21), we find that the identity (3.47) simplifies to

$$\bar{h}_{ij}(z) = z^{n_{ij}} \bar{h}_{ij}(z^{-1}) \tag{3.50}$$

Like $h_{ij}(z)$, the functions $\phi_1(z)$ and $\phi_3(z)$ are analytic for $|z| < 1/x$: so therefore is $\bar{h}_{ij}(z)$. From (3.50) it follows that $\bar{h}_{ij}(z)$ is also analytic for $|z| > x$, so it is entire. Further, when $z \to \infty$ it grows like $z^{n_{ij}}$, so it is a polynomial of degree n_{ij}, of the form

$$\bar{h}_{ij}(z) = a_0 + a_1 z + \cdots + a_1 z^{n_{ij}-1} + a_0 z^{n_{ij}} \tag{3.51}$$

where the coefficients $a_0, a_1, \ldots$ are independent of z.

In principle we can systematically calculate these polynomials. When $i = j = 1$, then $n_{ij} = 0$, so $\bar{h}_{11}(z)$ is a constant. This constant can be obtained from the orthogonality conditions for V, which imply that $h_{ij}(1/x) = \delta_{ij}$. Hence

$$h_{11}(z) = \tau^2 \phi_3(z) \tag{3.52}$$

where

$$\tau = \phi_3(x^{-1})^{-1/2} = \prod_{n=1}^{\infty} (1 + x^{2n})^{1/2} \left(\frac{1 - x^{2n-1}}{1 - x^{2n}} \right)^{n/2} \tag{3.53}$$

Now

$$\begin{aligned} \ln \phi_3(z) &= \sum_{n=1}^{\infty} \frac{z^n x^{2n}}{n(1 + x^n)^2 (1 + x^{2n})} \\ \ln \phi_1(z) &= -\sum_{n=1}^{\infty} \frac{z^n (x^n + x^{2n} + x^{3n})}{n(1 + x^n)^2 (1 + x^{2n})} \end{aligned} \tag{3.54}$$

We define

$$c_n = 1 + x + x^2 + \cdots + x^{n-1}, \qquad d_n = (1 + x^n)^{-1/2} \tag{3.55}$$

$$\rho_1 = 1, \qquad \rho_2 = x^{-1/2} d_1^2 d_2, \qquad \rho_3 = x^{-1} d_1^4 d_2^2 d_4 \tag{3.56}$$

$$\sigma_1 = 1, \qquad \sigma_2 = x^{1/2} d_1, \qquad \sigma_3 = x d_1^2 d_3, \qquad \sigma_4 = x^{3/2} d_1^3 d_2^2 d_3 \tag{3.57}$$

$$\tilde{V}_{ij} = V_{ij}/(\tau \rho_i \sigma_j), \qquad \tilde{h}_{ij}(z) = \bar{h}_{ij}(z)/(\tau^2 \sigma_i \sigma_j) \tag{3.58}$$

Then $\tilde{h}_{11}(z)=1$. Using (3.46) and the expansions (3.54), it follows (to within signs that are at our disposal) that

$$\tilde{V}_{11}=1, \qquad \tilde{V}_{21}=-x, \qquad \tilde{V}_{31}=x^2c_3 \tag{3.59}$$

Since $n_{12}=0$, $\bar{h}_{12}(z)$ is a constant and $h_{12}(z)$ is proportional to $\phi_1(z)$. Using (3.46), we obtain V_{12}, V_{22}, V_{32},... to within an overall normalization factor. This factor can then be obtained by constructing $\bar{h}_{22}(z)$ (which is linear, because $n_{22}=1$) and using the orthogonality relation $h_{22}(1/x)=1$. We find

$$\tilde{V}_{12}=1, \qquad \tilde{V}_{22}=c_3, \qquad \tilde{V}_{32}=-x^2c_5 \tag{3.60}$$

$$\tilde{h}_{12}(z)=1, \qquad \tilde{h}_{22}(z)=(1+z) \tag{3.61}$$

We can repeat this procedure for the third and fourth columns of V, using $n_{13}=1$ and $n_{14}=2$, and noting from the orthogonality condition that $h_{14}(1/x)=0$. It follows that $h_{13}(z)$ is proportional to $(1+z)\phi_1(z)$, and $h_{14}(z)$ to $(1-xz)(x-z)\phi_3(z)$. The proportionality constants can be evaluated from the relations $h_{33}(1/x)=h_{44}(1/x)=1$. We obtain

$$\tilde{V}_{13}=-x, \qquad \tilde{V}_{23}=c_5, \qquad \tilde{V}_{33}=c_3c_7 \tag{3.62}$$

$$\tilde{h}_{13}(z)=-x(1+z), \qquad \tilde{h}_{23}(z)=-(1-xz)(x-z) \tag{3.63}$$

$$\tilde{h}_{33}(z)=(1+z)(1+x^2z)(x^2+z) \tag{3.64}$$

$$\tilde{V}_{14}=x^2, \qquad \tilde{V}_{24}=c_3c_5, \qquad \tilde{V}_{34}=c_5c_7 \tag{3.65}$$

$$\tilde{h}_{14}(z)=x(1-xz)(x-z), \qquad \tilde{h}_{24}(z)=x^2(1+z)^2+(1+x)(1+x^3)z \tag{3.66}$$

$$\tilde{h}_{34}(z)=(1+z)[(1+x^2)c_5z-x^3(1+z^2)] \tag{3.67}$$

$$\tilde{h}_{44}(z)=x^4(1+z^4)+(1+x^2)[(1+x)^2(1+x^2)^2-2x^3](z+z^3) + (1+x^2+x^4)(1+x^4)z^2 \tag{3.68}$$

In principle one can extend this procedure indefinitely: for $j=5, 6,...$ one has to solve a finite set of homogeneous linear equations for $\tilde{V}_{1j},..., \tilde{V}_{jj}$ and the coefficients of the polynomials $\tilde{h}_{1j},..., \tilde{h}_{j-1,j}$. An overall normalization factor is then determined by the requirement $\tilde{h}_{jj}(1/x)=1$. Once $\tilde{h}_{1j},...,\tilde{h}_{j,j}$ are known, then $\tilde{V}_{ij}$ can be calculated for all values of i from (3.46).

A complication arises when one gets to those values of i or j for which the diagonal elements $\mathscr{C}_{ii}$ or $\mathscr{B}_{jj}$ of $\mathscr{C}$ or $\mathscr{B}$ are degenerate. This is because the original equation (3.30) is unchanged by the transformation $V \to Q^TVP$, where P (Q) is any orthogonal matrix that commutes with $\mathscr{B}$

($\mathscr{C}$). Even though $\mathscr{B}$ and $\mathscr{C}$ are diagonal, the degeneracy of their elements means that P and Q need only be block-diagonal. This introduces an arbitrariness into the calculation of V and $\tilde{h}$ from (3.30). [But this arbitrariness does not affect observables such as (3.45), the traces therein being unchanged by $V \to Q^T V P$.]

For instance, $\mathscr{C}_{44} = \mathscr{C}_{55} = 3\pi/K'$, which means that Q can have a two-by-two orthogonal block in rows and columns four and five. There is therefore one degree of freedom in choosing rows 4 and 5 of V. Indeed, substituting the above expressions for $\tilde{h}_{11},\ldots,\tilde{h}_{44}$ into (3.46) and equating the coefficients of z^3, to leading order in x (for x small), taking

$$\rho_4 = \rho_5 = x^{-3/2}$$

and requiring that $V \to 1$ as $x \to 0$, we find

$$\begin{gathered} \tilde{V}_{41} = (1-\Delta)x^4, \qquad \tilde{V}_{42} = (\Delta-2)x^3, \qquad \tilde{V}_{43} = -x, \qquad \tilde{V}_{44} = 1 \\ \tilde{V}_{51} = x^3, \qquad \tilde{V}_{52} = -x^2, \qquad \tilde{V}_{53} = x, \qquad \tilde{V}_{54} = \Delta x \end{gathered} \tag{3.69}$$

where Δ is an arbitrary constant of order unity.

This matrix functional method appears to extend easily to other models with the difference property. In 1982 P. A. Pearce and the author spent some time examining the corresponding equations for the eight-vertex model. We did obtain some elements of the corresponding V matrix (the matrices $\mathscr{B}$ and $\mathscr{C}$ are again simple direct sums). However, we were unsuccessful in our aim of proving the conjectured formula[57] for the spontaneous polarization.

The matrix V is related to the original matrix X_p by (3.29), and (3.46) is the specialization of (2.53) to the Ising model. For the Ising model itself there is probably little point in pursuing this calculation further, since V can be calculated more simply by spinor operators or the Clifford algebra.[58,59] The main point we wish to make here is that for the Ising model (and indeed for the eight-vertex model) the maxtrix functional relation (2.53) can be solved successively for the elements of X_p; and these elements are simple rational functions of the elliptic variables z and x. The question that this begs, and the one that motivates this paper, is whether this procedure can be extended to models without the difference property, notably the chiral Potts model. Further, does this lead to a natural parametrization of the model?

We shall be considering the three-state chiral Potts model in the next section, but we admit at once that we do not have the answers to these questions.

4. THREE-STATE CHIRAL POTTS MODEL

The simplest case of the solvable chiral Potts model that does *not* possess the difference property is when $N=3$. If k is fixed, then Eqs. (2.81) define an algebraic curve in homogeneous (a, b, c, d) space, of genus greater than one. There is no simple one-variable uniformizing substitution.

As reported in ref. 43, one *can* parametrize a, b, c, d in terms of hyperelliptic theta functions, but at the price of introducing two related variables s_1 and s_2. Here we write the nome q of ref. 43 as x. Remembering that $k'(1-k^2)^{1/2}$, we define x by Eq. (A9) of ref. 43:

$$\left(\frac{k'}{k}\right)^{1/3} = 3^{1/2}x^{1/6}\prod_{n=1}^{\infty}\left(\frac{1-x^{3n}}{1-x^n}\right)^2 \tag{4.1}$$

(k, k', x real, positive, and less than one), and a function $\Theta(s_1, s_2)$ by

$$\Theta(s_1, s_2) = \sum_{m,n} x^{m^2+mn+n^2}\exp[2\pi i(ms_1+ns_2)] \tag{4.2}$$

the sum being over all integers m, n.

Then a, b, c, d are given by Eq. (48) of ref. 43, where s_1 and s_2 are related by

$$\Theta(s_1+\tfrac{1}{2}\rho, s_2+\rho-\tfrac{1}{2}) = 0 \tag{4.3}$$

where ρ is positive pure imaginary and $x=e^{2\pi i\rho}$. If we set

$$z = e^{\pi i(2s_1+\rho)}, \qquad w = e^{2\pi i s_2}, \qquad t = z/w \tag{4.4}$$

then, using Eq. (24) of ref. 43, we find that

$$w = \prod_{n=1}^{\infty}\frac{(1-x^{2n-1}w/z)(1-x^{2n-1}z/w)(1-x^{6n-5}zw)(1-x^{6n-1}z^{-1}w^{-1})}{(1-x^{2n-2}z/w)(1-x^{2n}w/z)(1-x^{6n-4}z^{-1}w^{-1})(1-x^{6n-2}zw)} \tag{4.5}$$

or equivalently

$$\frac{z}{w} = \prod_{n=1}^{\infty}\frac{(1-x^{2n-2}/w)(1-x^{2n}w)(1-x^{6n-4}z^2/w)(1-x^{6n-2}w/z^2)}{(1-x^{2n-1}w)(1-x^{2n-1}/w)(1-x^{6n-5}w/z^2)(1-x^{6n-1}z^2/w)} \tag{4.6}$$

The variables z, w, t are associated with a rapidity p, so we can write them as z_p, w_p, t_p. Then

$$\begin{aligned} z_{Rp} &= xz_p, & w_{Rp} &= t_p, & t_{Rp} &= xw_p \\ z_{Sp} &= 1/(xz_p), & w_{Sp} &= 1/(xw_p), & t_{Sp} &= 1/t_p \end{aligned} \tag{4.7}$$

If we define

$$\phi(z) = z^{1/3} \prod_{n=1}^{\infty} \frac{(1-x^{3n-2}/z)(1-x^{3n-1}z)}{(1-x^{3n-2}z)(1-x^{3n-1}/z)} \tag{4.8}$$

then the Boltzmann weights (2.82) are[61]

$$\begin{aligned} W_{pr}(1) &= \phi(z_r/z_p)\,\phi(t_r/t_p), & W_{pr}(2) &= \phi(z_r/z_p)\,\phi(w_r/w_p) \\ \bar{W}_{pr}(1) &= \phi(xz_p/z_r)\,\phi(xw_p/t_r), & \bar{W}_{pr}(2) &= \phi(xz_p/z_r)\,\phi(t_p/w_r) \end{aligned} \tag{4.9}$$

If one defines f_{pr} as in,[8]

$$f_{pr} = \{\det_3(\bar{W}_{pr})/[W_{pr}(1)\,W_{pr}(2)]\}^{1/3} \tag{4.10}$$

then

$$f_{pr}/f_{rp} = \phi(z_p/z_r)\,\phi(w_p/w_r)\,\phi(t_p/t_r) \tag{4.11}$$

We have used this parametrization to solve perturbatively the original equations (2.3) and (2.4) in the low-temperature (small-x) limit. We expanded about the case when

$$\begin{gathered} z_p = O(1), \qquad z_q = O(x^{1/3}), \qquad z_r = O(x^{2/3}), \\ w_p \simeq 1 + z_p - O(1), \qquad w_q \simeq w_r \simeq 1 \end{gathered} \tag{4.12}$$

[this ensures that when x is small, all the Boltzmann weights $w_i(a, b)$ in (2.22) are small, provided $a \neq b$]. To relative first order in x it follows from Eq. (48) of ref. 43 (to within an overall normalization factor) that

$$\begin{gathered} a_p = e^{i\pi/6}(z_p w_p)^{1/3} x^{-1/6}(1-\omega x z_p)(1-\omega^2 x/z_p), \quad b_p = \omega y_p(1-\omega z_p) \\ c_p = 1-\omega^2 z_p, \quad d_p = e^{i\pi/6}(z_p w_p)^{1/3} x^{-1/6} y_p(1-x\omega/z_p)(1-x\omega^2 z_p) \end{gathered} \tag{4.13}$$

where

$$y_p = 1 + x\left(\frac{\omega^3}{w_p} + \frac{\omega z_p}{w_p} + \frac{1}{2}\right) + O(x^2)$$

and $\omega = e^{2\pi i/3}$. (Terms of order $x^{1/2}$ occur in Eq. (48) of ref. 43, but they can all be absorbed into the normalization factor.)

4.1. Leading-Order Solutions

As usual, we have in mind the situation when the matrices A_i are diagonal, with the rows and columns arranged so that the diagonal

elements of $A_1 \cdots A_6$ are decreasing. If we truncate the matrices to size n by n, then to leading order the CTM equations of Section 2 are satisfied. Further, the results can be matched (by appropriately arranging the rows and columns, and again only to leading order) with those obtained by truncating the lattices in Fig. 2, setting the boundary spins to zero.

The first lattice truncation is to set *all* spins other than the end spins a and b to zero. Then the A_i, F_i, G_i, H_i are all 3 by 3 matrices, e.g.,

$$\begin{aligned}(A_2)_{ab} &= [w_6(a)\, w_1(a)]^{1/2}\delta_{ab} \\ (F_2)_{ab} &= [w_6(a)\, w_1(b)\, w_2(a-b)]^{1/2} w_1(a)\, w_6(b) \\ &= w_2(a-b)^{1/2}(G_2)_{ab} = w_2(a-b)^{-1/2}(H_2)_{ab}\end{aligned} \tag{4.14}$$

Here we label the rows and columns simply by the end spins a and b, with values 0, 1, 2, rather than 1, 2, 3. The other matrices A_i, F_i, G_i, H_i can be obtained by cyclic permutations of the indices 1,..., 6.

At this level of truncation, the matrices A_i are already diagonal and Eqs. (2.24) and (2.25) are satisfied to leading order, with $\xi_i = \eta_i = 1$. Expressions (4.14) are, via (2.29) and (4.9), functions of the rapidities p, q, r. We use the notation $\mathrm{diag}(x, y, z)$ for the three-by-three diagonal matrix with diagonal elements x, y, z, and define

$$\alpha_p = (z_p^2/xw_p)^{1/3}, \qquad \beta_p = (w_p z_p)^{1/3}/x^{1/6} \tag{4.15}$$

$$J_p = \mathrm{diag}(1, \alpha_p, \beta_p) \tag{4.16}$$

Then to leading order

$$W_{pr}(1) = \alpha_r/\alpha_p, \qquad W_{pr}(2) = \beta_r/\beta_p \tag{4.17}$$

and one can verify that $A_2(p, q, r)$ is of the form (2.36), with

$$B_{pq} = (J_p J_q)^{1/2} \tag{4.18}$$

and from (2.42) and its G-analog

$$H_{pr} = J_p^{-1} \begin{pmatrix} 1 & x/z_r & x/z_r \\ z_p & 1 & t_p/w_r \\ z_p & xw_p/t_r & 1 \end{pmatrix} J_r \tag{4.19}$$

$$G_{pr} = J_r \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} J_p^{-1} \tag{4.20}$$

Because we are expanding about the case (4.12), we are no longer necessarily free to permute the rapidities ..., $R_{-1}r, p, q, r, Rp$,... in the formulas. Any given formula now has a domain of validity, which we should specify. Let us define "domains" $\mathscr{D}_{-2}, \mathscr{D}_{-1}, ..., \mathscr{D}_5$ by the statement that a *general* rapidity p belongs to $\mathscr{D}_i$ if

$$O(x^{(2i-1)/6}) \geqslant z_p \geqslant O(x^{(2i+1)/6}) \tag{4.21}$$

and $w_p \simeq (1+z_p)/(1+x/z_p)$. Further, $p \in \mathscr{D}_i$ iff $Rp \in \mathscr{D}_{i+3}$ or iff $Sp \in \mathscr{D}_{-i-3}$. Then the *particular* rapidities p, q, and r in (4.12) belong to $\mathscr{D}_0$, $\mathscr{D}_1$, and $\mathscr{D}_2$, respectively; while, for instance, $R^{-1}q \in \mathscr{D}_{-2}$ and $Rq \in \mathscr{D}_4$.

With these definitions, (4.18) is true if $p \in \mathscr{D}_i$ and $q \in \mathscr{D}_{i+1}$, for any integer i. Also, (4.19) and (4.20) are true if $p \in \mathscr{D}_i$ and $r \in \mathscr{D}_{i+2}$. The symmetry relations (2.38), (2.41), (2.66), and (2.67) are satisfied for all p, q, r, with

$$J_{Rp} = \tilde{M} J_p \mathscr{P}, \qquad J_{Sp} = Y^{-1} J_p^{-1} \tag{4.22}$$

$$\mathscr{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \tilde{M} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & x^{1/2} \\ 0 & x^{1/2} & 0 \end{pmatrix} \tag{4.23}$$

$$Y = M = \tilde{M}^2 = \mathrm{diag}(1, x, x) \tag{4.24}$$

Next we look at the factorization properties (2.47) and (2.49) and find that they can be satisfied. Define $\mathscr{N} = \mathrm{diag}(1, x, x)$. If $p \in \mathscr{D}_i$, then for $i = -1, 0, 1, 2$ we can choose

$$X_p = \begin{pmatrix} 1 & 1/z_p & 1/z_p \\ -xz_p - xt_p & 1 & 1-t_p \\ -xz_p & x(z_p^{-1} - 1 - z_p) & 1 \end{pmatrix} J_p \mathscr{N}$$

$$X_p^{-1} = \mathscr{N}^{-1} J_p^{-1} \begin{pmatrix} 1 & -1/z_p & -t_p/z_p \\ xz_p & 1 & t_p - 1 \\ xz_p & x(z_p - z_p^{-1}) & 1 \end{pmatrix}$$

For all p, X_p satisfies the relations (2.48), (2.68), (2.86), with

$$\tilde{L} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & x & x \\ 0 & x & -x^2 \end{pmatrix} \tag{4.26}$$

$$\Gamma = 1, \qquad L = \tilde{L}^2 \tag{4.27}$$

In particular, when $p \in \mathscr{D}_{-1}$, (2.86) can be verified by direct substitution of (4.25). Further, (2.86) can be regarded as a recursion relation, and can be used with (4.25) to calculate X_p for p in *any* domain $\mathscr{D}_i$.

We can go to the next level of truncation of the matrices by following the procedure in (2.55)–(2.62). We form the three-by-three matrices U_i^c as in (2.14), using (2..2), (4.14), and Z_N invariance. This gives

$$(U_2^c)_{ab} = [w_6(c-a)\, w_6(a)\, w_1(c-b)\, w_1(b)]^{1/2}\, w_2(a-b)\, w_1(a)\, w_6(b) \tag{4.28}$$

and similarly (by cyclic permutations of the indices 1,..., 6) for U_3^c,.... We define

$$\begin{aligned} J_p^{(0)} &= \mathrm{diag}(1, \alpha_p^3 \beta_p, \alpha_p \beta_p^3) \\ J_p^{(1)} &= \mathrm{diag}(\alpha_p, \alpha_p^3, \beta_p^4) \\ J_p^{(2)} &= \mathrm{diag}(\beta_p, \alpha_p^4, \beta_p^3) \end{aligned} \tag{4.29}$$

Then we can write U_2^c as

$$U_2^c = (J_p^{(c)} J_q^{(c)})^{-1/2}\, \tilde{U}_2^c (J_q^{(c)} J_r^{(c)})^{1/2} \tag{4.30}$$

where, for $p \in \mathscr{D}_i$, $q \in \mathscr{D}_{i+1}$, and $r \in \mathscr{D}_{i+2}$ (i any integer),

$$\begin{aligned} \tilde{U}_2^0 &= \begin{pmatrix} 1 & x^{3/2}/z_q z_r & x^{3/2}/z_q z_r \\ z_p z_q/x^{1/2} & 1 & t_p/w_r \\ z_p z_q/x^{1/2} & x w_p/t_r & 1 \end{pmatrix} \\ \tilde{U}_2^1 &= \begin{pmatrix} 1 & x/z_r & x/w_q z_r \\ z_p & 1 & t_p/w_q w_r \\ z_p w_q & x w_p w_q/t_r & 1 \end{pmatrix} \\ \tilde{U}_2^2 &= \begin{pmatrix} 1 & x^{3/2}/t_q z_r & x/z_r \\ z_p t_q/x^{1/2} & 1 & t_p t_q/x^{1/2} w_r \\ z_p & x^{3/2} w_p/t_q t_r & 1 \end{pmatrix} \end{aligned} \tag{4.31}$$

using the unorthodox convention that ab/cd means $ab/(cd)$.

To leading order, γ_{pqr} in (2.55) is one, so there must exist matrix functions $D_{pq}^{(c)}$ such that

$$U_2^c = (D_{pq}^{(c)})^{-1} D_{qr}^{(c)} \tag{4.32}$$

for $c = 0, 1, 2$.

We find that

$$D_{pq}^{(c)} = \tilde{D}_{pq}^{(c)} (J_p^{(c)} J_q^{(c)})^{1/2} \tag{4.33}$$

where if $p \in \mathscr{D}_i$, $q \in \mathscr{D}_{i+1}$, then for $i = -2, -1, 0, 1$ we can choose

$$\tilde{D}^{(0)}_{pq} = C_2^{-1} C_3^{-1} \begin{pmatrix} 1 & x/z_p z_q & x/z_p z_q \\ z_p z_q t_q - t_p z_q & 1 & -t_p - t_q \\ -z_p z_q & -xz_p + x/t_q & 1 \end{pmatrix} C_2 C_3$$

$$\tilde{D}^{(1)}_{pq} = C_1^{-1} \begin{pmatrix} 1 & x/z_q & x/w_p z_q \\ -z_p - t_q & 1 & (w_p w_q)^{-1} + x/2 \\ -z_p t_p (z_q + w_q) & -1 - 2xz_p z_q & 1 + t_p \end{pmatrix} \tag{4.34}$$

$$\tilde{D}^{(2)}_{pq} = C_3^{-1} \begin{pmatrix} 1 & x/t_p z_q & x/z_q \\ -z_p & 1/2 + x/z_p z_q & (1 + w_q)/2w_q \\ -t_p z_q & 1 & -t_p t_q - x/2 \end{pmatrix} C_2$$

$$(\tilde{D}^{(0)}_{pq})^{-1} = C_2^{-1} C_3^{-1} \begin{pmatrix} 1 & -x/z_p z_q & -x(1 + t_p + t_q)/z_p z_q \\ z_p z_q & 1 & t_p + t_q \\ z_p z_q & xz_p - x/t_q & 1 \end{pmatrix} C_2 C_3$$

$$(\tilde{D}^{(1)}_{pq})^{-1} = \begin{pmatrix} 1 & -x/z_q & -x/2z_p w_q \\ z_p & (1 + t_p)/2 & -(2w_p w_q)^{-1} - x/4 \\ z_p w_q & (1 + 2xz_p z_q)/2 & 1/2 \end{pmatrix} C_1 \tag{4.35}$$

$$(\tilde{D}^{(2)}_{pq})^{-1} = C_2^{-1} \begin{pmatrix} 1 & -x(1 + t_q)/z_q & -x(1 + w_p)/2z_p z_q w_q \\ z_p t_q & t_p t_q + x/2 & (1 + w_q)/2w_q \\ z_p & 1 & -x/z_p z_q - 1/2 \end{pmatrix} C_3$$

where

$$C_1 = \mathrm{diag}(1, \sqrt{2}, \sqrt{2}), \qquad C_2 = \mathrm{diag}(1, x^{1/2}, 1)$$
$$C_3 = \mathrm{diag}(1, 1, x^{1/2}) \tag{4.36}$$

Symmetries. If we define

$$\Omega_0 = \mathrm{diag}(1, x^2, x^2), \qquad \Omega_1 = \mathrm{diag}(x^{1/2}, x^{3/2}, x^2)$$
$$\Omega_2 = \mathrm{diag}(x^{1/2}, x^2, x^{3/2}) \tag{4.37}$$

then

$$J^{(c)}_{Rp} = \Omega_c \mathscr{P} J^{(-c)}_p \mathscr{P}, \qquad J^{(c)}_{Sp} = \Omega_c^{-2} J^{(c)^{-1}}_p \tag{4.38}$$

The matrix $D^{(c)}_{pq}$ can be regarded as the block c of a nine-by-nine block-diagonal matrix D_{pq} that satisfies the rotation, reflection, and half-rotation

symmetries (2.38), (2.66), and (2.86). Here the superscript $c = 0, 1, 2$ and is to be interpreted modulo 3, so the corresponding values of $-c$ are 0, 2, 1. The full nine-by-line matrices M and Y are block diagonal, while $\tilde{M}$ has the same structure as in (4.23), but with the elements therein replaced by 3 by 3 blocks (in block row c) as M_c, Y_c, and $\tilde{M}_c$, it follows that

$$\begin{aligned} \tilde{D}^{(c)T}_{pq} Y_c \tilde{D}^{(c)}_{Sq,Sp} &= \Omega_c^2 \\ \tilde{D}^{(c)}_{Rp,Rq} &= \tilde{M}_c \tilde{D}^{(-c)}_{pq} \mathscr{P} \Omega_c^{-1} \end{aligned} \tag{4.39}$$

from which we can deduce that

$$(\tilde{D}^{(c)^{-1}}_{pq})^T = Y_c \tilde{M}^{-1}_{-c} \tilde{D}^{(-c)}_{Rsq,RSp} \mathscr{P} \Omega_c^{-1} \tag{4.40}$$

By solving (4.32) for D_{pq}, we have verified that these symmetry relations are indeed satisfied at leading order, with

$$\tilde{M}_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & x^2 \\ 0 & x^2 & 2x^2 \end{pmatrix} \tag{4.41}$$

$$\begin{aligned} M_1 &= \tilde{M}_2 = \operatorname{diag}(x^{1/2}, \sqrt{2}x^{3/2}, x^2/\sqrt{2}) \\ Y_c &= M_c = \tilde{M}_c \tilde{M}_{-c} \end{aligned} \tag{4.42}$$

In particular, the RS symmetry (4.40) can be verified directly from the given results (4.34) and (4.35), while (4.39) can be used to obtain $\tilde{D}^{(c)}_{pq}$ for the case when $p \in \mathscr{D}_i$, $q \in \mathscr{D}_{i+1}$, and $i = 1, 2, 3, 4$. The results obtained match with (4.34) when $i = 1$, except for the element (3, 1) of $\tilde{D}^{(1)}_{pq}$: this is because this element is then small, and can only be obtained correctly to leading order by first calculating other elements to higher order. Similarly, the elements (3, 1) of $\tilde{D}^{(2)}_{pq}$ is not given correctly when $i = -2$ or 4.

We have started the procedure described at (2.20) for going from one truncation (in this case three by three) to a larger one. Because of (2.38) and (4.32), the eigenvalues of $U_1 \cdots U_6$ are those of M, and we have obtained nine of these, arranged in decreasing order, as

$$1, x, x, 2x^3, 2x^3, (\sqrt{2}+1)^2 x^4, x^4/2, x^4/2, (\sqrt{2}-1)^2 x^4 \tag{4.43}$$

Note that the first three eigenvalues are the same as those given in Eq. (4.24). The other six should be checked by going to higher truncations and observing that they do not change. In fact they do not, but as a general rule this procedure can only be trusted to give correctly the eigen-

values of current and next smallest order, so in this case we should consider only the first five eigenvalues 1, x, x, $2x^3$, $2x^3$.

The factor 2 in the last two of these is significant: it comes from the fact that the matrices $U_1^1,\ldots,U_6^1$ are not near-diagonal, in the sense that the eigenvalues of their product are the products of their diagonal elements (as is the case for the Ising model, and most of the previously solved models). If they were, then to leading order, $\tilde{M}_1$ would be diag$(x^{1/2}, x^{3/2}, x^2)$, and the diagonal elements of $\tilde{D}_{pq}^{(1)}$ would be constant (and hence could be chosen to be one).

To diagonalize $U_1^1 \cdots U_6^1$ to leading order, one must perform a full two-by-two diagonalization on the four lower right elements. This is the reason for the failure of the "wrong conjecture" of ref. 38. If it were true, then the full infinite matrices would be direct products of the 3 by 3 matrices in (4.18) and (4.24):

$$B_{pq} = (J_p J_q)^{1/2} \otimes (J_p J_q)^{3/2} \otimes (J_p J_q)^{5/2} \otimes \cdots$$

$$M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 0 \\ 0 & x^3 & 0 \\ 0 & 0 & x^3 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 & 0 \\ 0 & x^5 & 0 \\ 0 & 0 & x^5 \end{pmatrix} \otimes \cdots \tag{4.44}$$

This is not so, but what does appear to be true is that these formulas correctly give the order of the diagonal elements of B_{pq} and of the eigenvalues of M. For instance, (4.44) would give the largest nine eigenvalues of M to be 1, x, x, x^3, x^3, x^4, x^4, x^4, x^4, which agree in their order with (4.43). Further, the next-largest eigenvalues are correctly given as being of order x^5.

As with previous CTM calculations[46,44,47,39,40] it seems that if the largest eigenvalue neglected in a truncation is of order x^n, then solving the truncated equations (2.3)–(2.12) gives κ and $\langle s(a) \rangle$ correctly to order x^{n-1}. Thus in this case the 3 by 3 truncation should be accurate to order x^2, the 5 by 5 to order x^3, and the 9 by 9 to order x^4.

4.2. The Five-by-Five Truncation

The matrices M_0, M_1, M_2 are the same as the matrices $\mathcal{M}^0$, $\mathcal{M}^1$, $\mathcal{M}^2$ of (2.55)–(2.62). We can go to the next truncation in the way described after (2.62), selecting the largest five eigenvalues 1, x, x, $2x^3$, $2x^3$ of $\mathcal{M}^0$, $\mathcal{M}^1$, $\mathcal{M}^2$. Let us extend the definition (4.16) to a five-by-five diagonal matrix:

$$J_p = \text{diag}(1, \alpha_p, \beta_p, 2^{-1/2}\alpha_p, \beta_p) \tag{4.45}$$

and define

$$G_{pr} = J_r \tilde{G}_{pr} J_p^{-1}$$
$$H_{pr} = J_p^{-1} \tilde{H}_{pr} J_r \tag{4.46}$$

Then if $p \in \mathscr{D}_i$ and $r \in \mathscr{D}_{i+2}$, for $i = -4, -3, -2, -1$ we find

$$\tilde{G}_{pr} = \begin{pmatrix} 1 & 1 & 1 & x(1+xw_p)/z_p & 2x/z_p \\ 1 & 1 & 1 & -x/z_r & -2x^3 w_p - x/t_p z_r \\ 1 & 1 & 1 & -x/w_r - x/2z_r w_p & -x(1+t_r)/z_r \\ z_r + t_r & -z_p & -w_p - xz_p w_r/2 & 1 & 1/t_p + xz_r \\ z_r & -w_r^{-1} - z_p t_r/2 & -(z_p + w_p)/2 & t_r/2 + 1/2z_p & 1 \end{pmatrix} \tag{4.47}$$

$$\tilde{H}_{pr} = \begin{pmatrix} 1 & x/z_r & x/z_r & 2x^2(1+xw_p)/z_p z_r & 2x^2/z_p z_r \\ z_p & 1 & t_p/w_r & -2x/z_r & -2x^3 z_p/w_r - x/z_r w_r \\ z_p & xw_p/t_r & 1 & -x^2(2w_p + t_r^{-1})/z_r & -x(1+t_r)/z_r \\ z_p(z_r + t_r)/2 & -z_p/2 & -xz_p t_p/4 - z_p/2w_r & 1 & xt_p t_r/2 + 1/2w_r \\ z_p z_r & -xw_p(z_r^{-1} + z_p/2) & -(z_p + w_p)/2 & xw_p + x/t_p t_r & 1 \end{pmatrix} \tag{4.48}$$

while for $i = -1, 0, 1, 2$

$$\tilde{G}_{pr} = \begin{pmatrix} 1 & 1 & 1 & x/z_p & x(1+t_p)/z_p \\ 1 & 1 & 1 & -x(1+w_r)/2z_r & -x/t_r - x^2/2z_r t_p \\ 1 & 1 & 1 & -xt_p - x/2w_p z_r & -x/z_r \\ 2z_r & -z_p - t_p & -z_p w_r - 2x^2/t_r & 1 & w_r + x/z_p \\ z_r + xw_r & -t_p - z_p t_r/2 & -z_p & 1/2w_p + z_r/2 & 1 \end{pmatrix} \tag{4.49}$$

$$\tilde{H}_{pr} = \begin{pmatrix} 1 & x/z_r & x/z_r & 2x^2/z_p z_r & x^2(1+t_p)/z_p z_r \\ z_p & 1 & t_p/w_r & -x(1+w_r)/z_r & -xt_p/z_r - x^2/2z_r w_r \\ z_p & xw_p/t_r & 1 & +2x^2 z_p/t_r - x^2/z_r t_r & -x/z_r \\ z_p z_r & -(z_p + t_p)/2 & -x^2 t_p z_r - z_p t_p/2 & 1 & t_p/2 + x/2w_p w_r \\ z_p z_r + xz_p w_r & -xz_p(w_p/2 + 1/t_r) & -z_p & x/t_r + xw_p w_r & 1 \end{pmatrix} \tag{4.50}$$

We can verify that these matrices factorize as in (2.47), (2.49). In fact, for each element (i, j) of G_{pr} or H_{pr} we have calculated the first n_{ij} terms in an expansion in increasing powers of $x^{1/3}$, where

$$\{n_{ij}\} = \begin{pmatrix} 9 & 6 & 6 & 1 & 1 \\ 6 & 4 & 4 & 1 & 1 \\ 6 & 4 & 4 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \tag{4.51}$$

(The five-by-five truncation is accurate to at least two more terms than this.) For $p \in \mathscr{D}_{-2},\ldots,\mathscr{D}_1$, we find that to this accuracy we can take the matrix X_p in (2.47), (2.49) to be

$$X_p \times \tilde{X}_p J_p \mathscr{N} \tag{4.52}$$

where

$$\mathscr{N} = \begin{pmatrix} 1-x+4x^2 & 0 & 0 & 0 & 0 \\ 0 & x(1-2x) & 0 & 0 & 0 \\ 0 & 0 & x(1-5x/2) & 0 & 0 \\ 0 & 0 & 0 & 2x^2 & 0 \\ 0 & 0 & 0 & 0 & x^2 \end{pmatrix} \tag{4.53}$$

and, temporarily writing z_p and $\tilde{X}_p$ simply as z and $\tilde{X}$, we obtain the elements of $\tilde{X}$ as follows:

$$\tilde{X}_{11} = 1, \qquad \tilde{X}_{12} = z^{-1} - x/(1+z)$$

$$\tilde{X}_{13} = \frac{1}{z} + \frac{x^2}{z^2} - \frac{x^2}{1+xz}, \qquad \tilde{X}_{14} = \frac{x(1+2z)}{z^2(1+z)}, \qquad \tilde{X}_{15} = \frac{2x}{z^2}$$

$$\tilde{X}_{21} = -\frac{x(1-2x)\,z(2+z)}{1+z} + \frac{x^2(-1+z+3z^3+2z^3)}{(1+z)^3}, \qquad \tilde{X}_{22} = 1 + \frac{x}{z(1+z)}$$

$$\tilde{X}_{23} = \frac{1}{1+z} - \frac{x(2+3z+2z^2)}{(1+z)^3} - x^2 z, \qquad \tilde{X}_{24} = -\frac{1}{z}, \qquad \tilde{X}_{25} = -\frac{1}{z(1+z)}$$

$$\tilde{X}_{31} = \frac{(3x^2-x)z}{1+xz} - \frac{x^2}{1+z}, \qquad \tilde{X}_{32} = (1+2x)\left(-x + \frac{x}{z+x} - \frac{xz}{1+xz}\right) \tag{4.54}$$

$$\tilde{X}_{33} = 1, \qquad \tilde{X}_{34} = x - x/(2z^2), \qquad \tilde{X}_{35} = -z^{-1} - 1/(1+z)$$

$$\tilde{X}_{41} = -x^2 z^2 (3+z)/(1+z), \qquad \tilde{X}_{42} = xz(2+z)/(1+z)$$

$$\tilde{X}_{43} = xz/(1+z), \qquad \tilde{X}_{44} = 1, \qquad \tilde{X}_{45} = 1/(1+z)$$

$$\tilde{X}_{51} = -2^{1/2} x^2 z^2, \qquad \tilde{X}_{52} = 2^{1/2} x^2 (1-z^2)$$

$$\tilde{X}_{53} = 2^{1/2} xz, \qquad \tilde{X}_{54} = x(1-z^2)/(2^{1/2} z), \qquad \tilde{X}_{55} = 2^{1/2}$$

[The elements $\tilde{X}_{14}, \tilde{X}_{15}$ are not completely determined even to leading order at this stage of accuracy of the calculation: the above forms for them are consistent with those calculations that we have made. In particular, they satisfy the periodicity relations (2.86).]

To the appropriate orders of accuracy, the rotation symmetries (2.41), (2.48), and (2.86) (with F replaced by G or H) and the reflection symmetries (2.67)–(2.69) are satisfied, with $\Gamma = 1$,

$$\tilde{L} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & x-2x^2 & x-x^2 & 0 & 0 \\ 0 & x-x^2 & -x^2 & 0 & 0 \\ 0 & 0 & 0 & x^2 & 2^{1/2}x^2 \\ 0 & 0 & 0 & 2^{1/2}x^2 & 0 \end{pmatrix} \tag{4.55}$$

$$\tilde{M} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & x^{1/2}(1-x/2) & 0 & 0 \\ 0 & x^{1/2}(1-x/2) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2^{1/2}x^{3/2} \\ 0 & 0 & 0 & 2^{1/2}x^{3/2} & 0 \end{pmatrix} \tag{4.56}$$

$$Y = M = \tilde{M}^2 = \mathrm{diag}(1, x-x^2, x-x^2, 2x^3, 2x^3) \tag{4.57}$$

Together with (4.54), these symmetries can be used to obtain X_p for p in any domain $\mathscr{D}_i$.

If $p \in \mathscr{D}_i$ and $r \in \mathscr{D}_{i+2}$, then for $i = -1$ or 0, the scalar function η_{pr}, to order up to but not including x^3, is given by

$$\begin{aligned} \eta_{pr} = 1 &- \frac{x(3z+2)z'}{z(1+z)} - \frac{x^2(2z+1)}{z(1+z)} + \frac{xz'^2}{1+z} \\ &+ \frac{x^3(3z+2)}{z'} + \frac{x^3}{z(1+z)z'} - \frac{x^4(1+z+z^2)}{zz'^2} \\ &+ \frac{x^2z^2z'}{(1+z)^3} + \frac{x^2(1+11z+16z^2-2x^3)z'}{z^2(1+z)} - \frac{xz'^3}{z(1+z')} + x^2z'^2 + 2x^3z \end{aligned} \tag{4.58}$$

where here we have written z_p and z_r simply as z and z'. The symmetries (2.51) and (2.70) can be used to extend this result to all values of i. The constant $\alpha = \eta_{pp}$ in (2.52), (2.77), and (2.78) is

$$\alpha = 1 - 2x + 9x^2 + \cdots \tag{4.59}$$

4.3. The 17-by-17 Truncation

We have numerically developed series expansion solutions of the original equations (2.24)–(2.26). We worked in a representation in which $A_1,\ldots, A_6$ are diagonal. Following (2.39), we define M as the diagonal matrix similar to $A_1\cdots A_6$ (to within an overall scalar factor), so now $M=A_1\cdots A_6$.

Our aim was to obtain M: since this depends only on the nome x, and not on the rapidities, it was sufficient to assign numerical values to $z_p z_q$, z_r and to expand the matrices in powers of $x^{1/3}$, with numerical coefficients. We performed this calculation in Fortran. We retained the largest 17 diagonal elements, and accordingly truncated all the matrices to size 17 by 17. To leading order the diagonal elements of M (normalized so that $M_{11}=1$) are

$$1,\ x,\ x,\ 2x^3,\ 2x^3,\ (2^{1/2}+1)^2x^4,\ x^4/2,\ x^4/2,\ (2^{1/2}-1)^2x^4,\ 3x^5,\ 3x^5,$$
$$(3^{1/2}+1)^2x^6,\ 4x^6/3,\ 4x^6/3,\ (3^{1/2}-1)^2x^6,\ 9x^7/2,\ 9x^7/2 \tag{4.60}$$

and the next largest elements are of order x^8. [The corresponding values of the end spin $E(\lambda)$ are 0, 1, 2, 1, 2, 0, 1, 2, 0, 1, 2, 0, 1, 2, 0, 1, 2.] This truncation was therefore sufficient to obtain each element, and κ and $\langle\omega^{\pm a}\rangle$, to order $x^{23/3}$. We found

$$M_{11}=1$$
$$M_{22}=M_{33}=x-x^2+5x^3-21x^4+97x^5-479x^6+2449x^7+\cdots$$
$$M_{44}+M_{77}=2x^3-6x^4+34x^5-176x^6+941x^7+\cdots$$
$$M_{44}M_{77}=x^7-4x^8+34x^9-260x^{10}+\cdots$$
$$M_{55}=M_{44},\qquad M_{88}=M_{77}$$
$$M_{66}+M_{99}=6x^4-34x^5+201x^6-1184x^7+\cdots$$
$$M_{66}M_{99}=x^8-6x^9+77x^{10}-990x^{11}+\cdots$$
$$M_{10,10}+M_{13,13}+M_{16,16}=3x^5-10x^6+53x^7+\cdots \tag{4.61}$$
$$M_{10,10}M_{13,13}+M_{13,13}M_{16,16}+M_{16,16}M_{10,10}=4x^{11}-24x^{121}+\cdots$$
$$M_{10,10}M_{13,13}M_{16,16}=18x^{18}+\cdots$$
$$M_{11,11}=M_{10,10},\qquad M_{14,14}=M_{13,13},\qquad M_{17,17}=M_{16,16}$$
$$M_{12,12}+M_{15,15}=8x^6-28x^7+\cdots$$
$$M_{12,12}M_{15,15}=4x^{12}-40x^{13}+\cdots$$

The individual elements can be obtained to the available accuracy from these results, but the coefficients in their expansions are not all integers, or even rational. This suggests that it may not in fact be appropriate to completely diagonalize M: perhaps one should merely put it into some block-diagonal form.

Unlike the models with the difference property, we see that the eigenvalues of M are not simple powers of x, or even of M_{22}.

Let $\omega = e^{2\pi i/3}$ and take the function $s(a)$ in (2.13) to be ω^a. Then S is the diagonal matrix with elements 1, ω, ω^2, ω, ω^2, 1, ω, ω^2, 1, ω, ω^2, 1, ω, ω^2, 1, ω, ω^2, so

$$\langle \omega^a \rangle = (\zeta_0 + \omega\zeta_1 + \omega^2\zeta_2)/(\zeta_0 + \zeta_1 + \zeta_2) \tag{4.62}$$

where ζ_a is the sum of the eigenvalues of M for which $E(\lambda) = a$. Hence

$$\begin{aligned} \zeta_0 &= M_{11} + M_{66} + M_{99} + M_{12,12} + M_{15,15} \\ &= 1 + 6x^4 - 34x^5 + 209x^6 - 1212x^7 + \cdots \\ \zeta_1 &= M_{22} + M_{44} + M_{77} + M_{10,10} + M_{13,13} + M_{16,16} \\ &= x - x^2 + 7x^3 - 27x^4 + 134x^5 - 665x^6 + 3443x^7 + \cdots \end{aligned} \tag{4.63}$$

$\zeta_2 = \zeta_1$ and

$$\langle \omega^a \rangle = 1 - 3x + 9x^2 - 45x^3 + 231x^4 - 1224x^5 + 6669x^6 - 36978x^7 + \cdots \tag{4.64}$$

To this order this agrees with the conjecture (1.20) of Albertini *et al.*[15]:

$$\langle \omega^a \rangle = k^{2/9} \tag{4.65}$$

which had previously been made and verified to order x^6 by Howes *et al.*[2] (see also ref. 42). Like the eight-vertex model, the model is "Z invariant,"[48,60] so a one-site average such as $\langle \omega^a \rangle$ is the same for the square, hnoneycomb, and triangular lattices. Thus we have verified this conjecture to one more order.

Obviously there is much more to be done. For all previous models the eigenvalues of M (i.e., of $A_1 \cdots A_6$) have been obtained exactly. Can this be done for the chiral Potts model? Less ambitiously, can $\zeta_0, \zeta_1, \zeta_2$ be evaluated and the elegant conjecture (4.65) proved? Using (2.68) and (2.51), taking $\Gamma = 1$, and replacing p and r by $R^{-1}r$ and p, we can write Eq. (2.53) as

$$\eta_{r,Rp}(X^T_{RSr}X_p)_{\lambda\mu} = W_{pr}(a-b)(X^T_{RSp}X_r)_{\lambda\mu} \tag{4.66}$$

where $a = E(\lambda)$, $b = E(\mu)$. Note that $z_{RSp} = 1/z_p$, $w_{RSp} = 1/t_p$, $t_{RSp} = 1/w_p$, $\alpha_{RSp} = x^{-1/2}/\beta_p$, and $\beta_{RSp} = x^{-1/2}/\alpha_p$.

Together with the symmetry properties (2.48), (2.68), and (2.86), this equation defines X_p to within the transformation $X_p \to KX_pD^{-1}$ discussed after (2.87). However, even just the periodicity relation (2.86) (together with some insight into the analyticity properties of X_p) must impose severe restrictions on the form of X_p. [For the Ising model, each element of X_p is Laurent expandable in powers of $\exp(\pi p/2K')$ for all p: together with (2.86) or (2.48), just this simple observation implies that the eigenvalues of M are integer powers of $x^{1/2}$. The integers can then be obtained from low-temperature expansions, giving the diagonalized form of M.]

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